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Stepped-Impedance Transformers and Filter Prototypes*

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Summary—Quarter-wave transformers are widely used to obtain an impedance match within a specified tolerance between two lines of different characteristic impedances over a specified frequency band. This paper gives design formulas and extensive tables of designs, most of which were especially derived so that an integrated account could be presented for the first time. Numerous examples are given. Only homogeneous, synchronous transformers and filters are included in this paper, but a short bibliography on related topics is appended.

The theory is also applied to band-pass filters, by showing how to convert quarter-wave transformers into half-wave filter prototypes. The theoretical and numerical results presented are applicable to the design of impedance transformers, direct-coupled cavity filters, short-line low-pass filters, optical antireflection coatings and interference filters, acoustical transformers, branch-guide directional couplers, TEM-mode coupled-transmission-line directional couplers, and other circuits. These applications have been or will be dealt with in separate papers; this paper gives the basic theory and some of the numerical data required for these applications.

I. Introduction

THE OBJECTIVE of this paper[1] is to extend and consolidate the theory of the quarter-wave transformer, with two applications in mind: the first application is as an impedance-matching device or, literally, transformer; the second is as a prototype circuit, which shall serve as the basis for the design of various filters and directional couplers.

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[1] A more complete treatment is given in [1], on which this paper is based.

This paper is organized into nine parts, with the following purpose and content:

Section I is introductory. It also discusses applications, and gives a number of definitions.

Sections II and III deal with the performance characteristics of quarter-wave transformers and half-wave filters. In these parts the designer will find what *can* be done, not how to do it.

Sections IV to IX tell *how* to design quarter-wave transformers and half-wave filters. If simple general design formulas were available, solvable by nothing more complicated than a slide-rule, these parts would be much shorter.

Section IV gives exact formulas and numerical solutions for Chebyshev and maximally flat transformers of up to four sections.

Section V gives exact numerical solutions for maximally flat (but not Chebyshev) transformers of up to eight sections.

Section VI gives a first-order theory for Chebyshev and maximally flat transformers of up to eight sections, with explicit formulas and numerical tables. It also gives a general first-order formula, and refers to existing numerical tables published elsewhere which are suitable for up to 39 sections, and for relatively wide (but not narrow) bandwidths.

Section VII presents a modified first-order theory, accurate for larger transformer ratios than can be designed by the (unmodified) first-order theory of Section VI.

Sections VIII and IX apply primarily to prototypes for filters, since they are concerned with large impedance steps. They become exact only in the limit as the output-to-input impedance ratio R tends to infinity. Simple formulas are given for any number of sections, and previously published numerical tables on lumped-constant filters are referred to.

Sections VIII and IX complement Sections VI and VII which give exact results only in the limit as R tends to zero. It is pointed out that the dividing line between "small R" and "large R" is in the order of $[2/(\text{quarter-wave transformer bandwidth})]^{2n}$, where n is the number of sections. This determines whether the first-order theory of Sections VI and VII, or the formulas of Sections VIII and IX, are to be used. An example (Example 9) where R is in this borderline region is solved by both the "small R" and the "large R" approximations, and both methods give tolerably good results for most purposes.

Quarter-wave transformers have numerous applications besides being impedance transformers; an understanding of their behavior gives insight into many other physical situations not obviously connected with impedance transformations. The design equations and numerical tables have, moreover, been developed to the point where they can be used conveniently for the synthesis of circuits, many of which were previously difficult to design.

Circuits that can be designed using quarter-wave transformers as a prototype include: direct-coupled cavity filters [2]; impedance transformers [3]–[8]; optical interference filters and antireflection coatings [9], [10]; acoustical transformers [11], [12]; filters with quarter-wavelength resonators [13]; branch-guide couplers [14]; half-wave filters [15]; and short-line low-pass filters. It is intended to follow up this paper with others that will explain the design of some of these circuits, using the results and data published in this paper.

The insertion loss functions considered here are all for maximally flat or Chebyshev response in the pass band. It is of interest to note that occasionally other response shapes may be desirable. Thus TEM-mode coupled-transmission-line directional couplers are analytically equivalent to quarter-wave transformers [16], but require insertion loss functions with maximally flat or equal-ripple characteristics in the *stop*-band. Other insertion loss functions may be convenient for other applications. For instance, in optics refractive index corresponds to characteristic admittance, but is not as easily realized because of a limitation in available materials. The case when some refractive indexes (characteristic admittances) are given *a priori* leads to insertion loss functions different from those considered here [17].

As in the design of all microwave circuits, one must distinguish between the ideal circuits analyzed, and the actual circuits that have prompted the analysis, and which are the desired end product. To bring this out explicitly, we shall start with a list of definitions [18]:

Homogeneous transformer—a transformer in which the ratios of internal wavelengths and characteristic impedances at different positions along the direction of propagation are independent of frequency.

Inhomogeneous transformer—a transformer in which the ratios of internal wavelengths and characteristic impedances at different positions along the direction of propagation may change with frequency.

Quarter-wave transformer—a cascade of sections of lossless, uniform[2] transmission lines or media, each section being one-quarter (internal) wavelength long at a common frequency. *Note:* Homogeneous and inhomogeneous quarter-wave transformers are now defined by a combination of the above definitions. For instance, an *inhomogeneous quarter-wave transformer* is a quarter-wave transformer in which the ratios of internal wavelengths and characteristic impedances, taken between different sections, may change with frequency.

Ideal junction—the connection between two impedances or transmission lines, when the electrical effects of the connecting wires, or the junction discontinuities, can be neglected. (The junction effects may later be represented by equivalent reactances and transformers, or by positive and negative line lengths, etc.)

Ideal quarter-wave transformer—a quarter-wave transformer in which all of the junctions (of guides or media having different characteristic impedances) may be treated as ideal junctions.

Half-wave filter—a cascade of sections of lossless uniform transmission lines or media, each section being one-half (internal) wavelength long at a common frequency.

Synchronous tuning condition—a filter consisting of a series of discontinuities spaced along a transmission line is synchronously tuned if, at some fixed frequency in the pass band, the reflections from any pair of *successive* discontinuities are phased to give the maximum cancellation. (A quarter-wave transformer is a synchronously tuned circuit if its impedances form a monotone sequence. A half-wave filter is a synchronously tuned circuit if its impedances alternately increase and decrease at each step along its length.)

Synchronous frequency—the "fixed frequency" referred to in the previous definition will be called the *synchronous frequency.* (In the case of quarter-wave transformers, all sections are one-quarter wavelength long at the synchronous frequency; in the case of

[2] A uniform transmission line, medium, etc., is here defined as one in which the physical and electrical characteristics do not change with distance along the direction of propagation. This is a generalization of the IRE definition of *uniform waveguide* (See [19]).

half-wave filters, all sections are one-half wavelength long at the synchronous frequency. Short-line low-pass filters may also be derived from half-wave filters, with the synchronous frequency being thought of as zero frequency.)

The filters and transformers considered here are limited to *homogeneous*, *synchronous* types. For inhomogeneous [1], [7], [8], [18] or nonsynchronous [20] transformers, the additional bibliography should be consulted; this also lists references on the effect of dissipation losses and on the power handling capacity, which are not treated here.

Connection with Impedance Inverters

The realization of transmission-line discontinuities by impedance steps is equivalent to their realization by means of impedance inverters, popularized by Cohn. The main difference is that while impedance steps can be physically realized over a wide band of frequencies, at least for small steps, the impedance inverters can be physically realized over only small bandwidths. As far as using either circuit as a mathematical model, or prototype circuit, is concerned, they give equivalent results, as can be seen from Fig. 1.

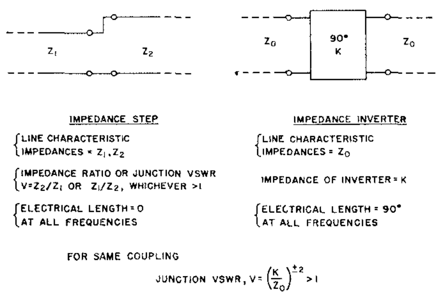

Fig. 1—Connection between impedance step and impedance inverter.

II. The Performance of Quarter-Wave Transformers

This section summarizes the relationships between the pass-band and stop-band attenuation; the fractional bandwidth w_q and the number of sections or resonators n. Although the expressions obtained hold exactly only for ideal transformers, they hold relatively accurately for real physical transformers and for certain filters, either without modification or after simple corrections have been applied.

A quarter-wave transformer is depicted in Fig. 2. Define the quarter-wave transformer fractional bandwidth w_q by

$$w_q = 2\left(\frac{\lambda_{g1} - \lambda_{g2}}{\lambda_{g1} + \lambda_{g2}}\right), \tag{1}$$

where λ_{g1} and λ_{g2} are the longest and shortest guide wavelengths, respectively, in the pass band of the quarter-wave transformer. The length, L, of each section (Fig. 2) is nominally one-quarter wavelength at center frequency and is given by

$$L = \frac{\lambda_{g1}\lambda_{g2}}{2(\lambda_{g1} + \lambda_{g2})} = \frac{\lambda_{g0}}{4}, \tag{2}$$

where the center frequency is defined as that frequency at which the guide wavelength λ_g is equal to λ_{g0}.

When the transmission line is nondispersive, the free-space wavelength λ may be used in (1) and (2), which then become

$$w_q = 2\left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\right) = 2\left(\frac{f_2 - f_1}{f_2 + f_1}\right) \tag{3}$$

and

$$L = \frac{\lambda_1\lambda_2}{2(\lambda_1 + \lambda_2)} = \frac{\lambda_0}{4}, \tag{4}$$

where f stands for frequency.

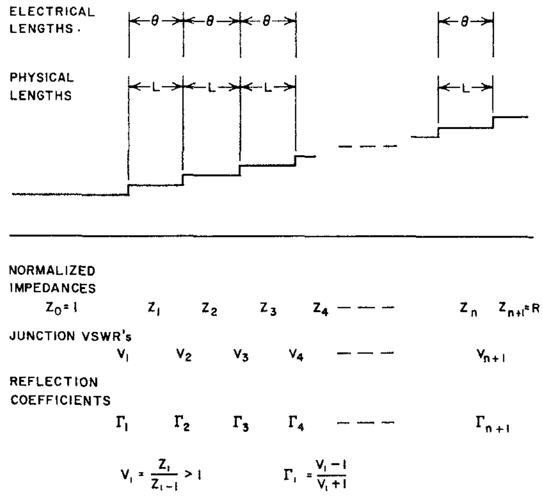

Fig. 2—Quarter-wave transformer notation.

The transducer loss ratio is defined as the ratio of P_{avail}, the available generator power, to P_L, the power actually delivered to the load. The "excess loss" $\mathcal{E}$ is herein defined by

$$\mathcal{E} = \frac{P_{\text{avail}}}{P_L} - 1. \tag{5}$$

For the maximally flat quarter-wave transformer of n sections and over-all impedance ratio R (Fig. 2), $\mathcal{E}$ is given by

$$\mathcal{E} = \frac{(R-1)^2}{4R}\cos^{2n}\theta = \mathcal{E}_\alpha \cos^{2n}\theta, \tag{6}$$

where

$$\theta = \frac{\pi}{2}\,\frac{\lambda_{g0}}{\lambda_g}, \tag{7}$$

λ_{g0} being the guide wavelength at band center, where

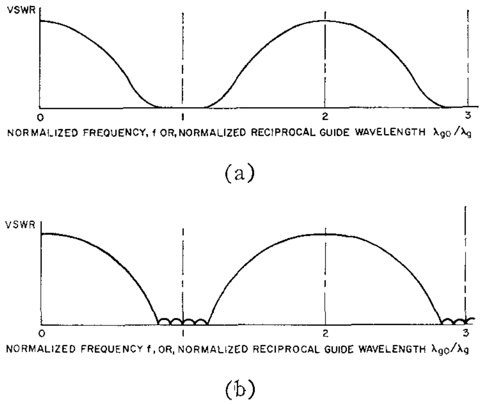

Fig. 3—Quarter-wave transformer characteristics. (a) Maximally flat. (b) Chebyshev.

$\theta=\pi/2$; and where

$$\mathcal{E}_\alpha = \frac{(R-1)^2}{4R} \tag{8}$$

is the greatest excess loss possible. (It occurs when θ is an integral multiple of π, since the sections then are an integral number of half-wavelengths long.)

The 3-db fractional bandwidth of the maximally flat quarter-wave transformer is given by

$$\omega_{q,3\text{ db}} = \frac{4}{\pi}\sin^{-1}\left[\frac{4R}{(R-1)^2}\right]^{1/2n}. \tag{9}$$

The fractional bandwidth of the maximally flat quarter-wave transformer between the points of x-db attenuation is given by

$$w_{q,x\text{ db}} = \frac{4}{\pi}\sin^{-1}\left\{\frac{4R[\text{antilog}(x/10)-1]}{(R-1)^2}\right\}^{1/2n}. \tag{10}$$

For the Chebyshev transformer of fractional bandwidth w_q,

$$\left.\begin{aligned}\mathcal{E} &= \frac{(R-1)^2}{4R}\,\frac{T_n^2(\cos\theta/\mu_0)}{T_n^2(1/\mu_0)}\\ &= \mathcal{E}_r T_n^2(\cos\theta/\mu_0)\end{aligned}\right\}, \tag{11}$$

where

$$\mu_0 = \sin\left(\frac{\pi w_q}{4}\right), \tag{12}$$

T_n is a Chebyshev polynomial (of the first kind) of order n, and where the quantity

$$\mathcal{E}_r = \frac{(R-1)^2}{4R}\cdot\frac{1}{T_n^2(1/\mu_0)} = \frac{\mathcal{E}_\alpha}{T_n^2(1/\mu_0)} \tag{13}$$

is the maximum excess loss in the pass band. [Compare also (18), below.] The shape of these response curves for maximally flat and Chebyshev quarter-wave transformers is shown in Fig. 3. Notice that the peak transducer loss ratio for any quarter-wave transformer is

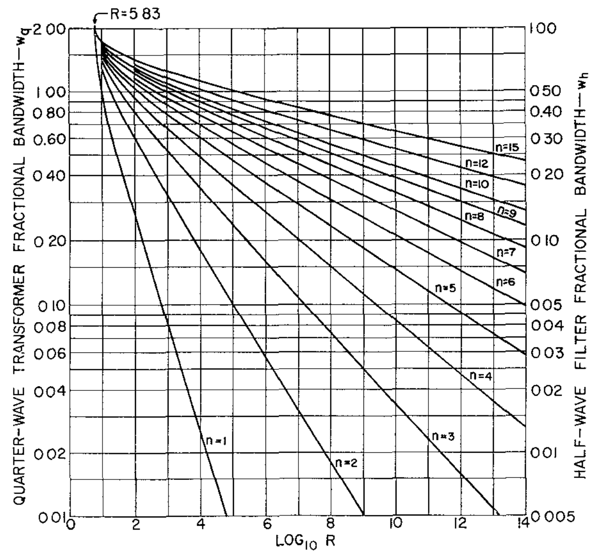

Fig. 4—3-db bandwidths of maximally flat transformers.

$$\frac{P_{\text{avail}}}{P_{\text{load}}} = \mathcal{E}_\alpha + 1 = \frac{(R+1)^2}{4R} \tag{14}$$

and is determined solely by the output-to-input impedance ratio, R.

For the maximally flat transformer, the 3-db fractional bandwidth, $w_{q,\,3\text{db}}$ is plotted against log R for $n=2$ to $n=15$ in Fig. 4. The attenuation given by (6) can also be determined from the corresponding lumped-constant low-pass prototype filter, which is available graphically in several references [21], [22] to $n=7$ and higher. If ω' is the frequency variable of the maximally flat, lumped-constant, low-pass prototype, and ω_1' is its band edge, then

$$\frac{\omega'}{\omega_1'} = \frac{\cos\theta}{\mu_0}, \tag{15}$$

where μ_0 is defined by (12), and w_q (which occurs in the definition of μ_0) is the fractional bandwidth of the maximally flat quarter-wave transformer between points of the same attenuation as the attenuation of the maximally flat low-pass filter at $\omega'=\omega_1'$. This enables one to turn existing charts of attenuation vs ω'/ω_1' (usually ω_1' corresponds to the 3-db points) into charts of attenuation vs $\cos\theta$ of the quarter-wave transformer, using (15).

For the Chebyshev transformer,

$$\frac{\mathcal{E}_\alpha}{\mathcal{E}_r} = T_n^2(1/\mu_0) = M(n, w_q), \tag{16}$$

where M is thus defined as a function of the number of sections n and the bandwidth w_q. It shows how much the pass-band tolerance increases when it is desired to improve the peak rejection. The function M in (16) is given in Table I for all fractional bandwidths, w_q, in

TABLE I

$$M(n, w_q) = T_n{}^2\left[\frac{1}{\sin(\pi w_q/4)}\right]$$

n \ w_q	0 1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
2	0.1049 *6	0.6517 *4	0.1274 *4	0.3978 *3	0.1601 *3	0.7575 *2	0.4001 *2	0.2293 *2	0.1400 *2	0.9000 *1
3	0.6795 *8	0.1052 *7	0.9094 *5	0.1584 *5	0.4036 *4	0.1306 *4	0.4972 *3	0.2130 *3	0.9966 *2	0.5000 *2
4	0.4402 *11	0.1699 *9	0.6491 *7	0.6313 *6	0.1020 *6	0.2265 *5	0.6246 *4	0.2013 *4	0.7291 *3	0.2890 *3
5	0.2851 *14	0.2742 *11	0.4634 *9	0.2517 *8	0.2578 *7	0.3930 *6	0.7852 *5	0.1906 *5	0.5353 *4	0.1682 *4
6	0.1847 *17	0.4427 *13	0.3308 *11	0.1003 *10	0.6516 *8	0.6819 *7	0.9872 *6	0.1806 *6	0.3933 *5	0.9801 *4
7	0.1196 *20	0.7148 *15	0.2361 *13	0.3999 *11	0.1646 *10	0.1183 *9	0.1241 *8	0.1710 *7	0.2890 *6	0.5712 *5
8	0.7751 *22	0.1154 *18	0.1685 *15	0.1594 *13	0.4162 *11	0.2052 *10	0.1560 *9	0.1620 *8	0.2123 *7	0.3329 *6
9	0.5021 *25	0.1863 *20	0.1203 *17	0.6355 *14	0.1052 *13	0.3561 *11	0.1961 *10	0.1535 *9	0.1560 *8	0.1940 *7
10	0.3252 *28	0.3008 *22	0.8590 *18	0.2533 *16	0.2658 *14	0.6178 *12	0.2466 *11	0.1454 *10	0.1146 *9	0.1131 *8
11	0.2107 *31	0.4856 *24	0.6132 *20	0.1010 *18	0.6720 *15	0.1072 *14	0.3100 *12	0.1377 *11	0.8422 *9	0.6592 *8
12	0.1356 *34	0.7840 *26	0.4377 *22	0.4026 *19	0.1698 *17	0.1860 *15	0.3898 *13	0.1304 *12	0.6188 *10	0.3842 *9
13	0.8842 *36	0.1266 *29	0.3124 *24	0.1605 *21	0.4292 *18	0.3227 *16	0.4901 *14	0.1235 *13	0.4547 *11	0.2239 *10
14	0.5728 *39	0.2044 *31	0.2230 *26	0.6397 *22	0.1084 *20	0.5598 *17	0.6161 *15	0.1170 *14	0.3340 *12	0.1305 *11
15	0.3710 *42	0.3299 *33	0.1592 *28	0 2550 *24	0.2742 *21	0.9712 *18	0.7746 *16	0.1108 *15	0.2454 *13	0.7607 *11

n \ w_q	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
2	0.6046 *1	0.4226 *1	0.3066 *1	0.2308 *1	0.1804 *1	0.1467 *1	0.1243 *1	0 1103 *1	0.1024 *1	1.0
3	0.2654 *2	0.1479 *2	0.8611 *1	0.5234 *1	0.3331 *1	0 2236 *1	0.1601 *1	0.1241 *1	0.1056 *1	1.0
4	0.1230 *3	0.5553 *2	0.2634 *2	0 1308 *2	0.6802 *1	0.3739 *1	0.2213 *1	0.1454 *1	0.1102 *1	1.0
5	0.5771 *3	0.2125 *3	0.8288 *2	0.3398 *2	0.1459 *2	0.6610 *1	0.3219 *1	0.1762 *1	0.1162 *1	1.0
6	0.2713 *4	0.8170 *3	0.2631 *3	0 8965 *2	0.3206 *2	0.1206 *2	0.4853 *1	0.2197 *1	0 1239 *1	1.0
7	0.1276 *5	0.3145 *4	0.8380 *3	0.2379 *3	0.7120 *2	0.2239 *2	0.7490 *1	0.2802 *1	0.1334 *1	1.0
8	0.6006 *5	0.1211 *5	0.2671 *4	0.6327 *3	0.1588 *3	0.4197 *2	0 1174 *2	0.3639 *1	0.1450 *1	1.0
9	0.2826 *6	0.4666 *5	0.8515 *4	0.1684 *4	0.3552 *3	0.7907 *2	0.1858 *2	0.4790 *1	0.1590 *1	1.0
10	0.1329 *7	0.1797 *6	0.2715 *5	0.4483 *4	0.7950 *3	0.1493 *3	0.2959 *2	0.6371 *1	0.1756 *1	1.0
11	0.6257 *7	0.6923 *6	0.8656 *5	0.1194 *5	0.1780 *4	0.2825 *3	0 4730 *2	0.8542 *1	0.1954 *1	1.0
12	0.2944 *8	0.2667 *7	0.2760 *6	0 3179 *5	0.3986 *4	0.5347 *3	0.7581 *2	0.1152 *2	0.2187 *1	1.0
13	0.1385 *9	0 1027 *8	0 8800 *6	0.8465 *5	0 8928 *4	0.1012 *4	0.1216 *3	0.1560 *2	0.2463 *1	1 0
14	0.6518 *9	0.3956 *8	0.2806 *7	0.2254 *6	0.1999 *5	0.1918 *4	0.1954 *3	0.2120 *2	0.2787 *1	1.0
15	0 3067 *10	0 1524 *9	0.8947 *7	0 6003 *6	0 4478 *5	0 3632 *4	0.3142 *3	0.2888 *2	0.3167 *1	1.0

* 4 means "multiply by 10^4," and so on

steps of 10 per cent, for $n=2$ to $n=15$. The smallest fractional bandwidth in Table I is $w_q=0.1$. For small bandwidths,

$$\frac{\mathcal{E}_\alpha}{\mathcal{E}_r} = T_n{}^2(1/\mu_0) \approx \frac{1}{4}\left(\frac{8}{\pi w_q}\right)^{2n}, \quad (w_q \text{ small}). \tag{17}$$

The attenuation given by (11) for the Chebyshev quarter-wave transformer can also be determined from graphs of the corresponding lumped-constant, low-pass, prototype filter [as already explained for the maximally flat case in connection with (15)] by using the same (15) except that now ω_1' is the Chebyshev (equal-ripple) band edge of the low-pass filter.[3]

The maximum VSWR may be worked out from Table I, using the relation

$$\mathcal{E}_r = \frac{(V_r - 1)^2}{4V_r} \tag{18}$$

[3] In [21], the lumped-constant characteristics for the Chebyshev filters are plotted against a frequency scale normalized with respect to the 3-db point and not the equal-ripple band edge. Since the curves in [21] are all plotted down to the equal-ripple band edge, this band-edge frequency can be read off and all frequencies divided by it, thus making $\omega'/\omega_1'=1$ at the equal-ripple band edge before applying (15).

where V_r is the ripple VSWR (maximum VSWR in the pass band), together with (8) and (16). The maximum VSWR for R less than 100 is also tabulated in [1] and [6].

Example 1: Determine the minimum number of sections for a transformer of impedance ratio $R=100$ to have a VSWR of less than 1.15 over a 100 per cent bandwidth ($w_q=1.0$).

From (18), for $V_r=1.15$,

$$\mathcal{E}_r = 0.00489 \tag{19}$$

and from (8), for $R=100$,

$$\mathcal{E}_\alpha = 24.5. \tag{20}$$

Hence, (16) gives

$$M(n, w_q) = T_n{}^2(1/\mu_0) = \frac{\mathcal{E}_\alpha}{\mathcal{E}_r} = 0.501 \times 10^4. \tag{21}$$

From Table I, in the column $w_q=1.0$, it is seen that this value of $M(n, w_q)$ falls between $n=5$ and $n=6$. Therefore, the transformer must have at least six sections. (See also Example 6.)

III. The Performance of Half-Wave Filters

The half-wave filter was defined in Section I. It is shown in Fig. 5. Its fractional bandwidth w_h is defined [compare (1)] by

$$w_h = 2\left(\frac{\lambda_{g1} - \lambda_{g2}}{\lambda_{g1} + \lambda_{g2}}\right) \tag{22}$$

and the length L' of each section [compare (2)] is

$$L' = \frac{\lambda_{g1}\lambda_{g2}}{\lambda_{g1} + \lambda_{g2}} = \frac{\lambda_{g0}}{2}, \tag{23}$$

where λ_{g1} and λ_{g2} are the longest and shortest wavelengths, respectively, in the pass band *of the half-wave filter*. This can be simplified for nondispersive lines by dropping the suffix "g," as in (3) and (4). A half-wave filter *with the same junction VSWR's* V_i (Figs. 2 and 5) as a quarter-wave transformer of bandwidth w_q has a bandwidth

$$w_h = \tfrac{1}{2} w_q \tag{24}$$

since its sections are twice as long and so twice as frequency-sensitive. The performance of a half-wave filter generally can be determined directly from the performance of the quarter-wave transformer with the same number of sections n and junction VSWR's V_i, by a linear scaling of the frequency axis by a scale-factor of 2. Compare Figs. 6 and 3. The quarter-wave transformer with the same n and V_i as the half-wave filter is herein called its *prototype* circuit.

In the case of the half-wave filter, R is the maximum VSWR, which is no longer the output-to-input impedance ratio, as for the quarter-wave transformer, but may generally be defined as the product of the junction VSWR's:

$$R = V_1 V_2 \cdots V_{n+1}. \tag{25}$$

This definition applies to both the quarter-wave transformer and the half-wave filter, as well as to filters whose prototype circuits they are. (In the latter case, the V_i are the individual discontinuity VSWR's.)

The equations corresponding to (6)–(18) will now be restated, wherever they differ, for the half-wave filter.

For the maximally flat half-wave filter of n sections

$$\mathcal{E} = \frac{(R-1)^2}{4R}\sin^{2n}\theta' = \mathcal{E}_a \sin^{2n}\theta', \tag{26}$$

where

$$\theta' = \pi\frac{\lambda_{g0}}{\lambda_g} = 2\theta \tag{27}$$

instead of (7), so that $\theta' = \pi$ (instead of $\theta = \pi/2$) at band center. The 3-db bandwidth of the maximally flat half-wave filter is

$$w_{h,\,3\,\mathrm{db}} = \tfrac{1}{2} w_{q,\,3\,\mathrm{db}} \tag{28}$$

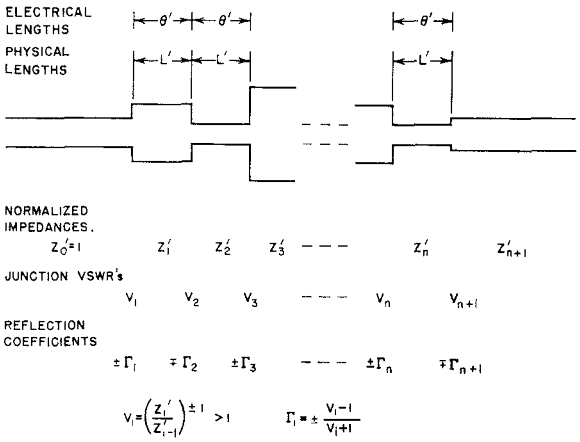

Fig. 5—Half-wave filter notation.

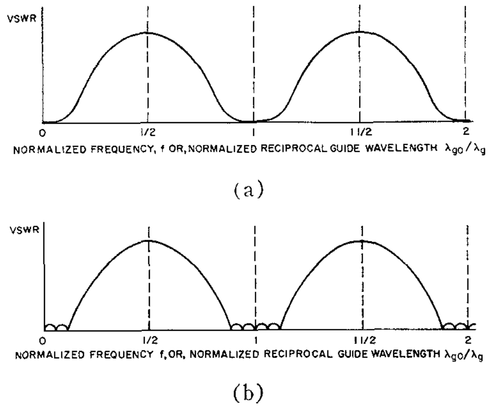

Fig. 6—Half-wave filter characteristics. (a) Maximally flat. (b) Chebyshev.

and the bandwidth between the points of x-db attenuation is

$$w_{h,x\,\mathrm{db}} = \tfrac{1}{2} w_{q,x\,\mathrm{db}} \tag{29}$$

which can be obtained from (9) and (10).

For the Chebyshev half-wave filter

$$\begin{aligned}\mathcal{E} &= \frac{(R-1)^2}{4R}\,\frac{T_n^2(\sin\theta'/\mu_0)}{T_n^2(1/\mu_0)} \\ &= \mathcal{E}_r T_n^2(\sin\theta'/\mu_0),\end{aligned} \tag{30}$$

where

$$\mu_0 = \sin\left(\frac{\pi w_q}{4}\right) = \sin\left(\frac{\pi w_h}{2}\right). \tag{31}$$

The quantities $\mathcal{E}_a$, $\mathcal{E}_r$, and the maximum transducer loss ratio are still given by (8), (13), and (14). For maximally flat half-wave filters, the graph of Fig. 4 can again be used, but with the right-hand scale.

The lumped-constant low-pass prototype filter graphs [21], [22] may again be used for both the maximally flat and Chebyshev half-wave filters by substituting

$$\frac{\omega'}{\omega_1'} = \frac{\sin\theta'}{\mu_0} \tag{32}$$

for (15), where μ_0 is given by (31).

Eq. (16) and Table I still apply, using (24) to convert between w_q and w_h.

Example 2: Find R for a half-wave filter of six sections having a Chebyshev fractional bandwidth of 60 per cent with a pass-band ripple of 1 db.

Here, $w_h = 0.6$, or $w_q = 1.2$. From (13),

$$\text{antilog}(0.1) - 1 = \frac{(R-1)^2}{4R} \frac{1}{T_6^2(1/\mu_0)} \tag{33}$$

and from Table I, for $w_q = 1.2$,

$$1.259 - 1 = \frac{(R-1)^2}{4R} \frac{1}{817}.$$

Hence, $R = 850$.

IV. Exact Chebyshev and Maximally Flat Solutions for up to Four Sections

Enough exact solutions will be presented to permit the solution of all intermediate cases by interpolation for Chebyshev and maximally flat transformers and filters having up to four sections.

The solutions were obtained from Collin's formulas [4]. With the notation of Fig. 1, they can be reduced to the expressions given below. The equations are first given for maximally flat transformers and then for Chebyshev transformers.

Maximally Flat Transformers for $n = 2$, 3, and 4

$$n = 2 \qquad \left.\begin{aligned} V_1 &= R^{1/4} \\ V_2 &= R^{1/2} \end{aligned}\right\} \tag{34}$$

$$n = 3 \qquad \left.\begin{aligned} &V_1^2 + 2R^{1/2}V_1 - \frac{2R^{1/2}}{V_1} - \frac{R}{V_1^2} = 0 \\ &V_2 = R^{1/2}/V_1 \end{aligned}\right\} \tag{35}$$

$$n = 4 \qquad \left.\begin{aligned} V_1 &= A_1 R^{1/8} \\ V_2 &= R^{1/4} \\ V_3 &= R^{1/4}/A_1^2 \\ \text{where} \quad & \left(\frac{1}{A_1^2} - A_1^2\right) = 2\left(\frac{R^{1/4}-1}{R^{1/4}+1}\right) \end{aligned}\right\} \tag{36}$$

Chebyshev Transformers for $n = 2$, 3, and 4

$$n = 2 \qquad \left.\begin{aligned} V_1^2 &= \sqrt{C^2 + R} + C \\ V_2 &= R/V_1^2 \\ \text{where} \quad C &= \frac{(R-1)\mu_0^2}{2(2-\mu_0^2)} \\ &\text{and } \mu_0 \text{ is given by (12).} \end{aligned}\right\} \tag{37}$$

$$n = 3 \qquad \left.\begin{aligned} &V_1^2 + 2\sqrt{R}\,V_1 - \frac{2\sqrt{R}}{V_1} - \frac{R}{V_1^2} = \frac{3\mu_0^2(R-1)}{4 - 3\mu_0^2} \\ &V_2 = R^{1/2}/V_1 \end{aligned}\right\} \tag{38}$$

$$n = 4 \qquad \left.\begin{aligned} V_1 &= \left\{R\left[B + \left(B^2 + \frac{A^2}{R}\right)^{1/2}\right]\right\}^{1/2} \\ V_2 &= \frac{1}{A} \\ V_3 &= \frac{A^2 R}{V_1^2} \\ \text{where} \\ A^2 &= \frac{1 - 1/R}{2t_1t_2} + \left[\frac{(1-1/R)^2}{4t_1^2t_2^2} + \frac{1}{R}\right]^{1/2} \\ B &= \frac{1}{2}\left(\frac{A}{A+1}\right)^2 \\ &\cdot \left[(t_1 + t_2)\left(A^2 - \frac{1}{A^2R}\right) - 2A + \frac{2}{AR}\right], \\ \text{and} \\ t_1 &= \frac{2\sqrt{2}}{(\sqrt{2}+1)\mu_0^2} - 1 \\ t_2 &= \frac{2\sqrt{2}}{(\sqrt{2}-1)\mu_0^2} - 1. \end{aligned}\right\} \tag{39}$$

A difference between typical quarter-wave transformers, and half-wave filters suitable for use as prototypes for microwave filters, is that, for the former, R is relatively small (usually less than 100) and only the pass-band performance is of interest; for the latter, R is relatively large, and the performance in both pass band and stop band is important. A set of tables for $n = 2$, 3, and 4, and for R from 1 to 100 has already been given in [6] (there they cover $w_q = 0$ to 1.2; they are extended to $w_q = 2.0$ in [1]).

The solutions of (34)–(39) for larger values of R are presented here in another set of tables (Tables II to V). They give the values of V_2 and V_3 for $n = 2$, 3, and 4. The remaining values of V are obtained from the symmetry relations

$$Z_i Z_{n+1-i} = R \tag{40}$$

(where the Z_i are normalized so that $Z_0 = 1$), or

$$V_i = V_{n+2-i} \tag{41}$$

or

$$\Gamma_i = \pm\,\Gamma_{n+2-i}. \tag{42}$$

Also

$$V_1 V_2 \cdots V_{n+1} = R \tag{43}$$

TABLE II

V_2 for Two-Section Quarter-Wave Transformers

w_q \ Log R	0	1	2	3	4	5	6	7	8	9	10	∞
0	1.0	3 1622	10.0000	31.6227	100.0000	316.2278	*	*	*	*	*	∞
0.2	1.0	3.1070	9.4056	26.0349	55.6931	76.0577	80.2075	80.6742	80.7215	80.7263	80.7267	80.7267
0.4	1.0	2.9446	7.8214	15.2942	19.2101	19.8657	19.9363	19.9434	19.9441	19.9442	19.9443	19.9443
0.6	1.0	2.6872	5.8168	8.1357	8.6395	8.6971	8.7030	8.7036	8.7037	8.7037	8.7037	8.7037
0.8	1.0	2.3592	4.0455	4.6882	4.7783	4.7878	4.7887	4 7888	4.7889	4 7889	4.7889	4.7889
1.0	1.0	2.0000	2.7937	2.9763	2.9976	2.9997	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
1.2	1.0	1.6569	1.9939	2.0491	2.0550	2.0556	2.0557	2.0557	2.0557	2.0557	2.0557	2.0557
1.4	1.0	1.3708	1.5000	1.5172	1.5190	1.5192	1 5192	1.5192	1 5192	1.5192	1.5192	1.5192
1.6	1.0	1.1635	1.2055	1.2105	1.2110	1.2111	1.2111	1.2111	1.2111	1.2111	1.2111	1.2111
1.8	1.0	1.0405	1.0491	1.0500	1.0501	1.0502	1.0502	1.0502	1 0502	1.0502	1.0502	1.0502
2.0	1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1 0000	1.0000	1.0000	1 0000	1.0000

TABLE III

V_2 for Three-Section Quarter-Wave Transformers

w_q \ Log R	0	1	2	3	4	5	6	7	8	9	10	∞
0	1.0	2.358	5.424	12.14	26.66	57.99	125.50	270.94	584.31	*	*	∞
0.2	1.0	2.344	5.337	11.68	24.46	47.69	79.82	101.7501	106.8067	107.3998	107.4610	107.4679
0.4	1.0	2.300	5.064	10.30	18.10	24.4154	26.1709	26.3912	26.4138	26.4161	26.4163	26.4164
0.6	1.0	2.222	4.5885	8.1080	10.6644	11.3276	11.4081	11.4163	11.4171	11.4172	11.4173	11.4173
0.8	1.0	2.103	3.9083	5.5671	6.1014	6.1728	6 1802	6.1809	6.1810	6.1810	6.1810	6.1810
1.0	1.0	1.9344	3.0919	3.6649	3.7630	3.7736	3.7747	3.7748	3.7748	3.7748	3.7748	3.7748
1.2	1.0	1.7158	2.3085	2.4686	2.4884	2.4904	2.4906	2.4907	2.4907	2.4907	2.4907	2.4907
1.4	1.0	1 4647	1.7022	1.7428	1 7472	1.7477	1.7477	1.7477	1.7477	1.7477	1.7477	1.7477
1.6	1.0	1.2269	1.2995	1.3089	1.3099	1.3100	1.3101	1.3101	1.3101	1.3101	1.3101	1.3101
1.8	1.0	1.0596	1.0731	1.0746	1.0747	1.0478	1.0748	1.0748	1.0748	1.0748	1.0748	1.0748
2.0	1.0	1.0000	1.0000	1.0000	1 0000	1.0000	1.0000	1.0000	1.0000	1 0000	1.0000	1.0000

TABLE IV

V_2 for Four-Section Quarter-Wave Transformers

w_q \ Log R	0	1	2	3	4	5	6	7	8	9	10	∞
0	1.0	1.7782	3.1622	5.6234	10.0000	17.7827	31.6227	56.2341	100.0000	177.8279	316.2277	∞
0.2	1.0	1.7781	3.1616	5.6200	9.9808	17.6752	31.0220	52.9321	82.9105	106.5498	113.2186	114.1604
0.4	1.0	1.7766	3.1524	5.5677	9.6904	16.1089	23.4997	27.3837	28.0996	28.1787	28.1867	28.1877
0.6	1.0	1.7698	3.1107	5.3364	8.4944	11.2447	12.1344	12.2543	12.2667	12.2680	12.2681	12.2682
0.8	1.0	1.7503	2.9928	4.7320	6.1880	6.6331	6.6915	6.6975	6.6981	6.6982	6.6982	6.6982
1.0	1.0	1.7054	2.7392	3.7126	4.0665	4.1171	4.1225	4.1230	4.1230	4.1231	4.1231	4.1231
1.2	1.0	1.6172	2.3157	2.6617	2.7224	2.7290	2.7297	2.7298	2.7298	2.7298	2.7298	2.7298
1.4	1.0	1.4676	1.8065	1.8903	1.9004	1.9014	1.9016	1.9016	1.9016	1.9016	1.9016	1.9016
1.6	1.0	1.2645	1.3725	1.3886	1.3904	1.3905	1.3906	1.3906	1.3906	1.3906	1.3906	1.3906
1.8	1.0	1 0768	1.0956	1.0978	1.9080	1.0980	1.0980	1.0980	1.0980	1.0980	1.0980	1.0980
2.0	1.0	1 0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1 0000	1 0000	1.0000	1.0000

TABLE V

V_3 for Four-Section Quarter-Wave Transformers

w_q \ Log R	0	1	2	3	4	5	6	7	8	9	10	∞
0	1.0	2.3448	5.2063	10.7833	21.1024	39.7366	73.0562	132.4190	238.0478	425.9232	760.0403	∞
0.2	1.0	2.3266	5.1045	10.3536	19.5265	34.4781	56.7950	85.7685	114.9996	132.4555	136.7239	137.7307
0.4	1.0	2.2728	4.8133	9.1941	15.6771	23.5245	30.2006	33.0054	33.4806	33.5323	33.5375	33.5382
0.6	1.0	2.1853	4.3689	7.6018	11.1871	13.5788	14.2420	14.3279	14.3367	14 3376	14.3377	14.3378
0.8	1.0	2.0664	3.8146	5.8599	7.2224	7.5865	7.6326	7.6374	7.6379	7 6380	7.6380	7.6380
1.0	1.0	1.9169	3.1842	4.1904	4.5121	4.5564	4.5610	4.5615	4.5615	4.5615	4.5615	4.5615
1.2	1.0	1.7347	2.5101	2.8558	2.9136	2.9199	2.9205	2.9206	2.9206	2.9206	2.9206	2.9206
1.4	1.0	1.5163	1.8739	1.9587	1.9688	1.9699	1.9700	1.9700	1.9700	1.9700	1.9700	1.9700
1.6	1.0	1 2767	1.3880	1.4045	1.4062	1.4064	1.4064	1.4064	1.4064	1.4064	1.4064	1.4064
1.8	1.0	1.0777	1.0967	1.0989	1.0991	1.0992	1.0992	1.0992	1.0992	1.0992	1.0992	1.0992
2.0	1.0	1 0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

which, for even n, reduces to

$$(V_1V_2 \cdots V_{n/2})^2 V_{(n/2)+1} = R \quad (44)$$

and for odd n, reduces to

$$(V_1V_2 \cdots V_{(n+1)/2})^2 = R. \quad (45)$$

Equations (40) to (45) hold for all values of n.

Tables II to V give the step VSWR's for R from 10 to ∞ in multiples of 10. Note that for Chebyshev transformers V_2, $V_3, \cdots,$ V_n and $V_1/(R)^{1/2} = V_{n+1}/(R)^{1/2}$ tend toward finite limits as R tends toward infinity, as can be seen from (34)–(39) for n up to 4, by letting R tend toward infinity. (For limiting values as R tends toward infinity and $n>4$, see Section IX.) The tables give fractional bandwidths, w_q, from 0 to 2.00 in steps of 0.20. [The greatest possible bandwidth is $w_q=2.00$, by definition, as can be seen from (1).]

When interpolating, it is generally sufficient to use only the two nearest values of V or Z. In that case, a linear interpolation on a log V or log Z against log R scale is preferable. Such interpolations, using only first differences, are most accurate for small R and for large R, and are least accurate in the neighborhood

$$R \sim \left(\frac{2}{w_q}\right)^{2(n-1)}. \quad (46)$$

In this region, second or higher order differences may be used (or a graphical interpolation may be more convenient) to achieve greater accuracy.

Example 3: Find the step VSWR's V_1, V_2, V_3, and V_4 for a three-section quarter-wave transformer of 80-per cent bandwidth and $R=200$. Also, find the maximum pass-band VSWR.

Here, $n=3$ and $w_q=0.8$. For $R=100$, from Table III,

$$V_2 = 3.9083$$
$$\therefore \log V_2 = 0.5920.$$

For $R=1000$,

$$V_2 = 5.5671$$
$$\therefore \log V_2 = 0.7456.$$

Now, for $R=200$,

$$\log R = 2.301.$$

Interpolating linearly,

$$\log V_2 = 0.5920 + 0.301(0.7456 - 0.5920)$$
$$= 0.6382$$
$$\therefore \quad V_2 = 4.347 = V_3 \text{ also.}$$

From (43) or (45),

$$(V_1V_2)^2 = R$$
$$\therefore \quad V_1 = V_4 = 2.086.$$

The maximum pass-band VSWR, V_r, is found from (8), (13), and Table I, which give $\varepsilon_r=0.23$, and then (18) determines the maximum pass-band VSWR, $V_r=2.5$.

V. Exact Maximally Flat Solutions for up to Eight Sections

Enough exact solutions will be presented to permit the solution of all intermediate cases by interpolation, for maximally flat transformers with up to eight sections.

The solutions were obtained by Riblet's method. This is a tedious procedure to carry out numerically; it requires high accuracy, especially for large values of R. In the limit as R becomes very large, approximate formulas adapted from Cohn's work on direct-coupled cavity filters [2] become quite accurate, and become exact in the limit, as R tends to infinity. This will be summarized in Section VIII. For our present purposes, it is sufficient to point out that, for maximally flat transformers, the ratios

$$\left.\begin{aligned} A_1 &= A_{n+1} = V_1/R^{1/2n} \\ A_i &= V_i/R^{1/n}, \quad i \neq 1 \text{ or } n+1 \end{aligned}\right\} \quad (47)$$

tend to finite limits as R tends to infinity. (See Section IX.)

Table VI gives the impedances Z_1 to Z_4 (Fig. 2) of maximally flat quarter-wave transformers of 5, 6, 7, and 8 sections for values of R up to 100. The impedances of maximally flat transformers of 2, 3, and 4 sections were already given in Tables II to V (case of $w_q=0$) and in [6]. The remaining impedances not given in these tables are determined from (40).

Table VII gives the A_i defined in (47) for maximally flat transformers of from 3 to 8 sections for values of R from 1 to ∞ in multiples of 10. The A_i change relatively little over the infinite range of R, thus permitting very accurate interpolation. The V_i are then obtained from (47), (41) and (43). The case $n=2$ is not tabulated, since the formulas in (34) are so simple.

VI. Approximate Design When R Is Small

First-Order Theory

Exact numerical Chebyshev solutions for $n>4$, corresponding to the maximally flat solutions up to $n=8$ in Section V, have not yet been computed. When the output-to-input impedance ratio R approaches unity, the reflection coefficients of the impedance steps approach zero, and a first-order theory is adequate. The first-order theory assumes that each discontinuity (impedance step) sets up a reflected wave of small amplitude, and that these reflected waves pass through the other small discontinuities without setting up further second-order reflections. This theory holds for "small R" as defined by

$$R < \left(\frac{2}{w_q}\right)^{n/2} \quad (48)$$

and can be useful even when R approaches $(2/w_q)^n$, particularly for large bandwidths.

TABLE VI

IMPEDANCES OF MAXIMALLY FLAT TRANSFORMERS

R	n=5		n=6			n=7			n=8			
	Z_1	Z_2	Z_1	Z_2	Z_3	Z_1	Z_2	Z_3	Z_1	Z_2	Z_3	Z_4
1.5	1.01277	1.07904	1.00636	1.04540	1.14960	1.00318	1.02570	1.09628	1.00158	1.01438	1.06041	1.15872
2.0	1.02201	1.13908	1.01096	1.07904	1.26929	1.00547	1.04448	1.17039	1.00273	1.02481	1.10571	1.28658
2.5	1.02931	1.18816	1.01458	1.10608	1.37082	1.00727	1.05944	1.23157	1.00363	1.03307	1.14243	1.39558
3.0	1.03539	1.23002	1.01759	1.12884	1.45995	1.00878	1.07195	1.28415	1.00438	1.03997	1.17355	1.49162
3.5	1.04061	1.26672	1.02018	1.14861	1.53996	1.01007	1.08275	1.33055	1.00503	1.04590	1.20071	1.57813
4.0	1.04521	1.29954	1.02246	1.16613	1.61292	1.01121	1.09229	1.37227	1.00560	1.05114	1.22490	1.65722
4.5	1.04932	1.32931	1.02450	1.18191	1.68026	1.01223	1.10085	1.41030	1.00611	1.05583	1.24678	1.73039
5.0	1.05305	1.35663	1.02635	1.19631	1.74297	1.01315	1.10863	1.44534	1.00658	1.06009	1.26681	1.79870
6.0	1.05962	1.40549	1.02961	1.22186	1.85731	1.01479	1.12240	1.50837	1.00740	1.06762	1.30252	1.92356
7.0	1.06530	1.44845	1.03243	1.24413	1.96010	1.01620	1.13436	1.56414	1.00812	1.07414	1.33381	2.03617
8.0	1.07032	1.48696	1.03493	1.26395	2.05396	1.01746	1.14496	1.61440	1.00875	1.07992	1.36177	2.13926
9.0	1.07482	1.52196	1.03717	1.28186	2.14066	1.01859	1.15451	1.66032	1.00932	1.08513	1.38714	2.23474
10.0	1.07892	1.55413	1.03921	1.29822	2.22148	1.01962	1.16322	1.70270	1.00984	1.08987	1.41041	2.32393
15.0	1.09531	1.68600	1.04740	1.36450	2.56378	1.02375	1.19830	1.87818	1.01194	1.10895	1.50543	2.70350
20.0	1.10760	1.78804	1.05356	1.41497	2.84017	1.02688	1.22484	2.01581	1.01354	1.12335	1.57860	3.01198
25.0	1.11753	1.87251	1.05855	1.45628	3.07621	1.02942	1.24645	2.13089	1.01484	1.13507	1.63889	3.27666
30.0	1.12592	1.94524	1.06277	1.49152	3.28448	1.03158	1.26482	2.23080	1.01594	1.14502	1.69087	3.51111
35.0	1.13322	2.00950	1.06646	1.52243	3.47223	1.03347	1.28087	2.31965	1.01692	1.15371	1.73661	3.72308
40.0	1.13969	2.06729	1.06973	1.55006	3.64407	1.03515	1.29518	2.40004	1.01778	1.16146	1.77770	3.91762
45.0	1.14552	2.12000	1.07268	1.57510	3.80311	1.03667	1.30812	2.47372	1.01857	1.16845	1.81513	4.09813
50.0	1.15084	2.16856	1.07538	1.59807	3.95162	1.03805	1.31996	2.54192	1.01928	1.17485	1.84958	4.26701
60.0	1.16027	2.25588	1.08017	1.63911	4.22331	1.04052	1.34106	2.66530	1.02056	1.18624	1.91145	4.57684
70.0	1.16847	2.33312	1.08434	1.67513	4.46845	1.04268	1.35951	2.77519	1.02168	1.19620	1.96609	4.85724
80.0	1.17575	2.40267	1.08805	1.70736	4.69297	1.00460	1.37597	2.87473	1.02269	1.20507	2.01523	5.11474
90.0	1.18230	2.46613	1.09139	1.73661	4.90095	1.04634	1.39087	2.96605	1.02359	1.21310	2.06003	5.35379
100.0	1.18828	2.52464	1.09444	1.76343	5.09522	1.04793	1.40450	3.05064	1.02442	1.22043	2.10129	5.57761

TABLE VII

A_i OF MAXIMALLY FLAT TRANSFORMERS

$$\left(\begin{array}{l} A_1 = A_{n+1} = V_1/R^{1/2n} \\ A_i = V_i/R^{1/n}, \text{ when } i \neq 1, n+1 \end{array}\right)$$

Log R	n=3	n=4*	n=5		n=6			n=7			n=8			
	A_1	A_1	A_1	A_2	A_2	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3	A^4
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	0.9135	0.8708	0.8570	0.9088	0.8577	0.8510	1.1658	0.8649	0.8210	1.0534	0.8744	0.8093	0.9704	1.2355
2	0.8557	0.7793	0.7497	0.8458	0.7456	0.7478	1.3411	0.7541	0.6941	1.1250	0.7682	0.6699	0.9682	1.4926
3	0.8239	0.7221	0.6755	0.8084	0.6619	0.6837	1.5107	0.6664	0.6110	1.2147	0.6803	0.5736	0.9966	1.7432
4	0.8080	0.6883	0.6263	0.7873	0.6013	0.6451	1.6629	0.5987	0.5578	1.3131	0.6090	0.5084	1.0468	1.9665
5	0.8004	0.6689	0.5943	0.7753	0.5582	0.6217	1.7911	0.5473	0.5238	1.4103	0.5519	0.4645	1.1088	2.1524
6	0.7968	0.6579	0.5738	0.7684	0.5281	0.6073	1.8934	0.5087	0.5016	1.4992	0.5069	0.4348	1.1745	2.2997
7	0.7951	0.6516	0.5607	0.7643	0.5071	0.5983	1.9717	0.4801	0.4871	1.5760	0.4717	0.4144	1.2384	2.4125
8	0.7943	0.6481	0.5523	0.7618	0.4926	0.5924	2.0296	0.4590	0.4773	1.6394	0.4444	0.4003	1.2969	2.4976
9	—	0.6461	0.5471	0.7603	0.4827	0.5886	2.0716	0.4436	0.4707	1.6900	0.4234	0.3904	1.3481	2.5610
10	—	0.6450	0.5437	0.7594	0.4758	0.5861	2.1013	0.4324	0.4661	1.7293	0.4074	0.3834	1.3914	2.6078
11	—	—	0.5416	0.7588	0.4712	0.5845	2.1222	0.4242	0.4630	1.7593	0.3952	0.3784	1.4270	2.6423
12	—	—	0.5403	0.7584	0.4680	0.5833	2.1366	0.4183	0.4607	1.7817	0.3860	0.3747	1.4557	2.6681
∞	0.7937	0.6436	0.5380	0.7579	0.4612	0.5810	2.1684	0.4031	0.4553	1.8433	0.3578	0.3646	1.5538	2.7430

* For $n=4$, $A_2=1.0000$.

Denote the reflection coefficients of an n-section transformer or filter by

$$\Gamma_i, \quad \text{where} \quad i = 1, 2, \cdots, n+1$$

to give a Chebyshev response of bandwidth, w_q. Let

$$c = \cos\left(\frac{\pi w_q}{4}\right). \tag{49}$$

The quantity c is related to μ_0 of (12) by

$$c^2 + \mu_0^2 = 1. \tag{50}$$

Then, for n-section Chebyshev transformers, the following ratio formulas relate the reflection coefficients up to $n=8$.

For $n=2$,

$$\Gamma_1:\Gamma_2=1:2c^2. \tag{51}$$

For $n=3$,

$$\Gamma_1:\Gamma_2=1:3c^2. \tag{52}$$

For $n=4$,

$$\Gamma_1:\Gamma_2:\Gamma_3=1:4c^2:2c^2(2+c^2). \tag{53}$$

For $n=5$,

$$\Gamma_1:\Gamma_2:\Gamma_3=1:5c^2:5c^2(1+c^2). \tag{54}$$

TABLE VIII

TABLE OF Γ_i/Γ_1

Bandwidth, w_q	$n=2$	$n=3$	$n=4$		$n=5$		$n=6$			$n=7$			$n=8$			
	$i=2$	$i=2$	$i=2$	$i=3$	$i=2$	$i=3$	$i=2$	$i=3$	$i=4$	$i=2$	$i=3$	$i=4$	$i=2$	$i=3$	$i=4$	$i=5$
0.0	2.0000	3.0000	4 0000	6.0000	5.0000	10.0000	6.0000	15.0000	20.0000	7.0000	21.0000	35.0000	8.0000	28.0000	56.0000	70.0000
0.2	1.9511	2.9266	3.9021	5.8054	4.8776	9.6359	5 8532	14.4181	19 1298	6.8287	20.1519	33.3120	7.8042	26.8373	53.1111	66.1559
0.4	1.8090	2 7135	3 6180	5.2543	4.5225	8.6132	5.4270	12.7903	16.7247	6 3316	17.7855	28.6925	7.2361	23.5988	45 2566	55.7879
0.6	1.5878	2.3817	3.1756	4.4361	3.9695	7.1208	4.7634	10.4357	13.3273	5.5572	14.3810	22.2954	6.3511	18.9564	34.5254	41.8439
0.8	1.3090	1.9635	2.6180	3.4748	3.2725	5.4144	3.9271	7.7825	9.6284	4 5816	10.5789	15.5402	5.2361	13.8037	23.4303	27.7539
1.0	1.0000	1.5000	2.0000	2.5000	2.5000	3.7500	3.0000	5.2500	6.2500	3 5000	7.0000	9.6250	4.0000	9.0000	14.0000	16.1250
1.2	0.6910	1.0365	1.3820	1.6207	1.7275	2.3243	2 0729	3.1472	3.5878	2.4184	4 0895	5.2138	2.7639	5.1512	7.2434	8.0793
1.4	0 4122	0.6183	0.8244	0.9094	1.0305	1.2429	1.2366	1.6190	1.7639	1.4428	2 0375	2 3961	1.6489	2.4985	3.1483	3.3919
1.6	0.1910	0.2865	0.3820	0.4002	0.4775	0.5231	0.5730	0.6550	0 6841	0.6684	0.7961	0.8660	0.7639	0.9463	1.0697	1.1133
1.8	0.0489	0.0734	0.0979	0 0991	0.1224	0.1254	0.1468	0.1522	0.1540	0.1713	0.1797	0.1840	0.1958	0.2078	0.2152	0.2177
2.0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

For $n=6$,

$$\Gamma_1:\Gamma_2:\Gamma_3:\Gamma_4=1:6c^2:3c^2(2+3c^2):2c^2(3+6c^2+c^4). \quad (55)$$

For $n=7$,

$$\Gamma_1:\Gamma_2:\Gamma_3:\Gamma_4=1:7c^2:7c^2(1+2c^2):7c^2(1+3c^2+c^4). \quad (56)$$

For $n=8$,

$$\Gamma_1:\Gamma_2:\Gamma_3:\Gamma_4:\Gamma_5=1:8c^2:4c^2(2+5c^2):8c^2(1+4c^2+2c^4)$$
$$:2c^2(4+18c^2+12c^4+c^6). \quad (57)$$

Table VIII tabulates the Γ_i/Γ_1 for all fractional bandwidths in steps of 20 per cent in w_q, for transformers of up to eight sections. The Γ's are obtained from the appropriate one of the above equations, or from Table VIII together with (42) and the specified value of R. (See Example 4.) When $w_q=0$ (maximally flat case), the Γ's reduce to the binomial coefficients. (A general formula for any n will be given below.)

Range of Validity of First-Order Theory

For a transformer of given bandwidth, as R increases from unity on up, the Γ_i all increase at the same rate according to the first-order theory, keeping the ratios Γ_i/Γ_1 constant. Eventually one of the Γ_i would exceed unity, resulting in a physically impossible situation, and showing that the first-order theory has been pushed too far. To extend the range of validity of the first-order theory, it has been found advantageous to substitute $\log V_i$ for Γ_i. This substitution [23], which appears to be due to W. W. Hansen [3], might be expected to work better, since, first, $\log V_i$ will do just as well as Γ_i when the Γ_i are small compared to unity, as then

$$\left.\begin{aligned}\log V_i &= \log\frac{1+\Gamma_i}{1-\Gamma_i}\\ &\approx \text{constant} \times \Gamma_i\end{aligned}\right\} \quad (58)$$

and, second, $\log V_i$ can increase indefinitely with increasing $\log R$ and still be physically realizable.

The first-order theory generally gives good results in the pass band when $\log V_i$ is substituted for Γ_i, provided that R is "small" as defined by (48). (Compare end of Section IX.)

Example 4: Design a six-section quarter-wave transformer of 40-per cent bandwidth for an impedance ratio of $R=10$. [This transformer will have a VSWR less than 1.005 in the pass band, from (8) and (18) and Table I.]

Here $(2/w_q)^{n/2}=125$, which is appreciably greater than $R=10$. Therefore, we can proceed by the first-order theory. From Table VIII,

$$\log V_1:\log V_2:\log V_3:\log V_4 = 1:5.4270:12.7903:16.7247$$

$$\therefore \frac{\log V_1}{\log R} = \frac{\log V_1}{\sum_{i=1}^{7}\log V_i} = \frac{1}{55.1593} = 0.01813.$$

Since $\log R=\log 10=1$,

$$\therefore V_1 = V_7 = \text{antilog } (0.01813) = 1.0426$$
$$V_2 = V_6 = \text{antilog } (5.4270 \times 0.01813) = 1.254$$
$$V_3 = V_5 = \text{antilog } (12.7903 \times 0.01813) = 1.705$$

and

$$V_4 = \text{antilog } (16.7247 \times 0.01813) = 2.010.$$

Hence,

$$\begin{aligned} Z_1 &= V_1 &&= 1.0426\\ Z_2 &= V_2Z_1 &&= 1.308\\ Z_3 &= V_3Z_2 &&= 2.228\\ Z_4 &= V_4Z_3 &&= 4.485\\ Z_5 &= V_5Z_4 &&= 7.65\\ Z_6 &= V_6Z_5 &&= 9.60\\ R = Z_7 &= V_7Z_6 &&= 10.00.\end{aligned}$$

Relation to Dolph-Chebyshev Antenna Arrays

When R is small, numerical solutions of certain cases up to $n=39$ may be obtained through the use of existing antenna tables. The first-order Chebyshev transformer problem is mathematically the same as Dolph's solu-

TABLE IX
TRANSFORMER-ARRAY CORRESPONDENCES

Chebyshev Transformer	Dolph-Chebyshev Array
First-order theory	Optical diffraction theory
Synchronous tuning	Uniform phase (or linear phase taper)
Frequency	Angle in space
Transformer length	Array length
Pass band	Side-lobe region
Stop band	Main lobe
Reflection coefficient	Radiation field
Number of steps $(n+1)$	Number of elements
$M(n, w_q)$	Side-lobe ratio
$10 \log_{10} M$	Side-lobe level in db
$\log V_i$	Element currents, I_i

tion [24] of the linear array, and the correspondences shown in Table IX may be set up.

The calculation of transformers from tables of graphs or array solutions is best illustrated by an example.

Example 5: Design a transformer of impedance ratio $R=5$ to have a maximum VSWR, V_r, of less than 1.02 over a 140-per cent bandwidth ($w_q=1.4$).

It is first necessary to determine the minimum number of sections. This is easly done as in Example 1, using Table I, and is determined to be $n=11$.

Applying the test of (48),

$$\left(\frac{2}{w_q}\right)^{n/2} = 50,$$

whereas R is only 5, and so we may expect the first-order theory to furnish an accurate design.

The most extensive tables of array solutions are contained in [25]. (Some additional tables are given in [26].) We first work out M from (8), (18), and (16), and find $M=8000$. Hence the side-lobe level is

$$10 \log_{10} M = 39.0 \text{ db}.$$

From Table II in [25], the currents of an $n+1=12$ element array of side-lobe level 39 db are respectively proportional to 3.249, 6.894, 12.21, 18.00, 22.96, 25.82, 25.82, 22.96, 18.00, 12.21, 6.894, and 3.249. Their sum is 178.266. Since the currents are to be proportional to $\log V_i$, and since $R=5$, $\log R=0.69897$, we multiply these currents by $0.69897/178.266=0.003921$ to obtain the $\log V_i$. Taking antilogarithms yields the V_i and, finally, multiplying yields the Z_i. [Compare Ex. 4.] Thus Z_0 through R are respectively found to be 1.0, 1.0298, 1.09585, 1.2236, 1.4395, 1.7709, 2.2360, 2.8233, 3.4735, 4.0861, 4.5626, 4.8552, and 5.0000. The response of this transformer is plotted in Fig. 7, and is found to satisfy the specifications almost perfectly.

In antenna theory, one is usually not interested in side-lobe ratios in excess of 40 db; this is as far as the antenna tables take us. Only fairly large bandwidths can be calculated with this 40-db limit. For example, Table I shows that for $n=2$ this limits us to $w_q>0.18$;

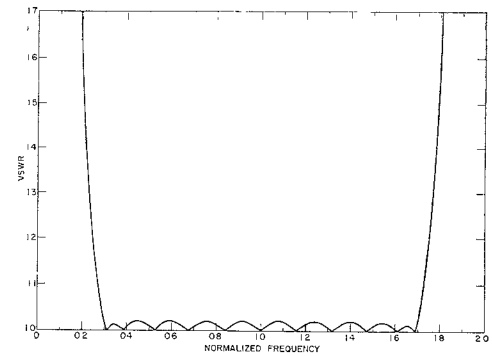

Fig. 7—Analyzed performance of transformer designed in Example 5.

for $n=4$, to $w_q>0.67$; for $n=8$, to $w_q>1.21$; and for $n=12$, to $w_q>1.52$. A general formula for all cases has been given by G. J. Van der Maas [27] which becomes, when adapted to the transformer,

$$\frac{\Gamma_i}{\Gamma_1} = \frac{n}{n+1-i}\sum_{r=0}^{i-2}\binom{n+1-i}{r+1}\binom{i-2}{r}c^{2(r+1)} \tag{59}$$

for $2 \leq i \leq (n/2)+1$, where c is given by (49), and $\binom{a}{b}$ are the binomial coefficients

$$\binom{a}{b} = \frac{a!}{b!(a-b)!}. \tag{60}$$

VII. APPROXIMATE DESIGN FOR UP TO MODERATELY LARGE R

Modified First-Order Theory

In Section VI, a first-order theory was presented which held for "small" values of R as defined by (48). In Section VIII, there will be presented formulas that hold for "large" values of R as defined by (73). This leaves an intermediate region without explicit formulas. Since exact numerical solutions for maximally flat transformers of up to eight sections have been tabulated (Tables VI and VII), these might be used in conjunction with either the "small R" or the "large R" theories to extend the one upward or the other downward in R, and so obtain more accurate solutions for Chebyshev

TABLE X

TABLE OF γ_i

Bandwidth, w_q	$n=2$		$n=3$		$n=4$			$n=5$		
	$i=1$	$i=2$	$i=1$	$i=2$	$i=1$	$i=2$	$i=3$	$i=1$	$i=2$	$i=3$
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.2	1.01237	0.98762	1.01869	0.99376	1.02501	0.99992	0.99176	1.03135	1.00611	0.99380
0.4	1.05014	0.94985	1.07715	0.97428	1 10418	0.99873	0.96695	1.13188	1.02379	0.97491
0.6	1.11488	0.88511	1.18283	0.93905	1.25124	0.99336	0.92510	1.32337	1.05062	0.94234
0.8	1.20882	0.79117	1 34975	0.88341	1.49381	0.97770	0.86512	1.65171	1.08104	0.89430
1.0	1.33333	0.66666	1.60000	0.80000	1.88235	0.94117	0.78431	2.20689	1.10344	0.82758
1.2	1.48643	0.51356	1.96415	0.67861	2.50599	0.86571	0.67690	3.16718	1.09426	0.73614
1.4	1.65823	0.34176	2.47172	0.50942	3.51015	0.72344	0.53202	4.88788	1.00739	0.60751
1.6	1.82565	0.17434	3.10921	0 29692	5.05657	0.48290	0.33727	7.99760	0.76377	0.41835
1.8	1.95226	0.04773	3.72647	0.09117	6.97198	0.17063	0.11515	12.82256	0.31389	0.16079
2.0	2.0	0	4.0	0	8.0	0	0	16.0	0	0

Bandwidth, w_q	$n=6$				$n=7$				$n=8$				
	$i=1$	$i=2$	$i=3$	$i=4$	$i=1$	$i=2$	$i=3$	$i=4$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.2	1.03774	1.01235	0.99748	0.99258	1.04417	1.01861	1.00200	0.99381	1.05063	1.02492	1.00701	0.99643	0.99294
0.4	1 16027	1.04946	0.98935	0 97026	1.18937	1.07581	1.00731	0.97503	1.21921	1.10279	1.02757	0.98531	0.97167
0.6	1.39965	1 11118	0.97375	0.93268	1.48033	1.17521	1.01374	0.94298	1.56565	1.24295	1.05997	0.96526	0.93590
0.8	1.82608	1.19520	0.94743	0.87911	2.01888	1.32138	1.01702	0.89639	2.23202	1.46088	1.10036	0.93387	0.88496
1 0	2.58585	1.29292	0.90505	0.80808	3.02958	1.51479	1.00986	0.83313	3.54939	1.77469	1.14087	0.88734	0.81762
1.2	3.99301	1.37951	0.83778	0.71630	5.03077	1.73806	0.97968	0.74941	6.33721	2.18942	1.16586	0.81969	0.73143
1.4	6.75454	1.39211	0.72904	0.59571	9.30719	1.91834	0.90301	0.63717	12.81069	2.64044	1.14312	0.72021	0.62075
1.6	12.45111	1.18908	0.54369	0.42589	19.31633	1.83488	0.72848	0.47546	29.51655	2.81846	0.99755	0.56381	0.46943
1.8	23 25581	0.56899	0.23596	0.17906	41.69381	1.02030	0.35677	0.21919	74.08908	1.81333	0.54984	0.28471	0.23041
2 0	32 0	0	0	0	64 0	0	0	0	128.0	0	0	0	0

transformers with R in this intermediate region. This idea is applied here to the first-order ("small R") theory only, as will be explained. It extends the range of the first-order theory from the upper limit given by (48) up to "moderately large" values of R as defined by

$$R < \left(\frac{2}{w_q}\right)^n \tag{61}$$

and gives acceptable results even up to the square of this limit,

$$R < \left(\frac{2}{w_q}\right)^{2n}. \tag{62}$$

[Compare (73).] Of course, when R is less than specified by (48), there is no need to go beyond the simpler first-order theory of Section VI.

The first step in the proposed modification of the first-order theory is to form ratios of the Γ_i, which will be denoted by γ_i, with the property that

$$\left\{\frac{\Gamma_i}{\sum_{i=1}^{n+1} \Gamma_i}\right\}_{\substack{\text{Chebyshev}\\ \text{transformer}}} = \gamma_i \left\{\frac{\Gamma_i}{\sum_{i=1}^{n+1} \Gamma_i}\right\}_{\substack{\text{maximally flat}\\ \text{transformer}}} \tag{63a}$$

The γ_i are functions of n (the same n for both transformers) and w_q (the bandwidth of the desired Chebyshev transformer). The substitution of $\log V_i$ for Γ_i will again be used, and therefore $\sum_{i=1}^{n+1} \Gamma_i$ is replaced by $\log R$, according to (43). If now we choose R to be the same for both the Chebyshev transformer and the corresponding maximally flat transformer, then (63a) reduces to

$$(\log V_i)_{\substack{\text{Chebyshev}\\ \text{transformer}}} = \gamma_i (\log V_i)_{\substack{\text{maximally flat}\\ \text{transformer}}} \tag{63b}$$

The modification to the first-order theory now consists in using the *exact* $\log V_i$ of the maximally flat transformer where these are known (Tables VI and VII). The γ_i could be obtained from (63) and Table VIII, but are tabulated for greater convenience in Table X. The numbers in the first row of this table are, by definition, all unity. The application of this table is illustrated by an example given below.

Range of Validity of the Modified First-Order Theory

The analyzed performance of a first-order design, modified as explained above, and illustrated in Example 6, agrees well with the predicted performance, provided that R satisfies (61) or at least (62). (In this regard, compare the end of Section IX.)

As a rough but useful guide, the first-order modification of the exact maximally flat design generally gives good results when the pass-band maximum VSWR is less than or equal to $(1+w_q^2)$, where w_q is the equal-ripple quarter-wave transformer bandwidth (1). By definition, it becomes exact when $w_q=0$.

Example 6: In Example 1 it was shown that a quarter-wave transformer of impedance ratio $R=100$, fractional bandwidth, $w_q=1.00$, and maximum pass-band VSWR of less than 1.15 must have at least six sections ($n=6$). Calculate the normalized line impedances Z_i of this quarter-wave transformer. Predict the maximum pass-band VSWR, V_r. Then, also find the bandwidth w_h and normalized line impedances, Z_i', of the corresponding half-wave filter.

First, check that R is small enough for the transformer to be solved by a first-order theory. Using (48),

$$\left(\frac{2}{w_q}\right)^{n/2} = 2^3 = 8. \tag{64}$$

Therefore the unmodified first-order theory would not be expected to give good results, since $R=100$ is considerably greater than 8. Using (61) and (62),

$$\left.\begin{aligned}\left(\frac{2}{w_q}\right)^{n} &= 64\\ \left(\frac{2}{w_q}\right)^{2n} &= 2048\end{aligned}\right\}. \tag{65}$$

Therefore the modified first-order theory should work quite well, although we may expect noticeable but not excessive deviation from the desired performance since $R=100$ is slightly greater than $(2/w_q)^n=64$.

From Table VI and Fig. 2, or from Table VII and (47), it can be seen that a maximally flat transformer of six sections with $R=100$ has

$$\left.\begin{aligned}V_1 = V_7 &= 1.094 \quad \therefore \log V_1 = 0.0391\\ V_2 = V_6 &= 1.610 \quad \therefore \log V_2 = 0.2068\\ V_3 = V_5 &= 2.892 \quad \therefore \log V_3 = 0.4612\\ V_4 &= 3.851 \quad \therefore \log V_4 = 0.5856\end{aligned}\right\}. \tag{66}$$

The log VSWR's of the required 100-per cent bandwidth transformer are now obtained, according to (63b), multiplying the log V's in (66) by the appropriate values of γ in Table IX:

$$\left.\begin{aligned}\log V_1 &= 0.0391 \times 2.586 = 0.1011\\ \log V_2 &= 0.2068 \times 1.293 = 0.2679\\ \log V_3 &= 0.4612 \times 0.905 = 0.4170\\ \log V_4 &= 0.5856 \times 0.808 = 0.4733\end{aligned}\right\} \tag{67}$$

$$\left.\begin{aligned}\therefore V_1 = V_7 &= 1.262\\ V_2 = V_6 &= 1.853\\ V_3 = V_5 &= 2.612\\ V_4 &= 2.974\end{aligned}\right\}. \tag{68}$$

Now this product $V_1V_2 \cdots V_7$ equals 105.4, instead of 100. It is therefore necessary to scale the V_i slightly downward, so that their product reduces to exactly 100. The preferred procedure is to reduce V_1 and V_7 by a factor of $(100/105.4)^{1/12}$ while reducing $V_2, \cdots, V_6$ by a factor of $(100/105.4)^{1/6}$. It can be shown [see Example 8 and (74)] that this type of scaling, where V_1 and V_{n+1} are scaled by the square root of the scaling factor[4] for $V_2, \cdots, V_n$, has as its principal effect a slight increase in bandwidth while leaving the pass-band ripple almost unaffected. Since the approximate designs generally fall slightly short in bandwidth, while coming very close to, or even improving on, the specified pass-band ripple, this method of scaling is preferable. Subtracting 0.0038 from log V_1 and 0.0076 from the remaining log V_i in (67) gives the new V_i

$$\left.\begin{aligned}V_1 = V_7 &= 1.251\\ V_2 = V_6 &= 1.821\\ V_3 = V_5 &= 2.566\\ V_4 &= 2.922\end{aligned}\right\} \tag{69}$$

and for the corresponding normalized line impedances of the quarter-wave transformer (Fig. 2),

$$\left.\begin{aligned}Z_0 &= 1.0\\ Z_1 = V_1 &= 1.251\\ Z_2 = Z_1V_2 &= 2.280\\ Z_3 = Z_2V_3 &= 5.850\\ Z_4 = Z_3V_4 &= 17.10\\ Z_5 = Z_4V_5 &= 43.91\\ Z_6 = Z_5V_6 &= 79.94\\ R = Z_6V_7 &= 100.00\end{aligned}\right\}. \tag{70}$$

We note in passing that the product of the VSWR's before reduction was 105.4 instead of the specified 100. If the discrepancy between these two numbers exceeds about 5 to 10 per cent, the predicted performance will usually not be realized very closely.

The maximum insertion loss and VSWR in the pass band predicted from (16) and Table I are

$$\mathcal{E}_r = 0.0025, \quad \text{or} \quad 0.011 \text{ db}$$

Therefore, by (18),

$$V_r = 1.106. \tag{71}$$

The computed plot of V against normalized frequency, f, of this transformer (or against λ_{g0}/λ_g if the transformer is dispersive) is shown in Fig. 8. The bandwidth is 95 per cent (compared to 100 per cent predicted) for a maximum pass-band VSWR of 1.11. (Notice that the response has equal ripple heights with a maximum VSWR of 1.065 over an 86-per cent bandwidth.)

[4] In general, if R' and R are respectively the trial and desired impedance ratios, then for an nth-order transformer, the scaling factor is ${}^{(1/n)}\sqrt{R/R'}$ for $V_1, V_3, \cdots V_n$, and ${}^{(1/2n)}\sqrt{R/R'}$ for V_1 and V_{n+1}.

The bandwidth w_h of the half-wave filter for a maximum VSWR of 1.11 will be just half the corresponding bandwidth of the quarter-wave transformer, namely 47.5 per cent. Its normalized line impedances are (see Fig. 5):

$$\left.\begin{aligned} Z_0' &= 1.0 \quad \text{(input)} \\ Z_1' &= V_1 = 1.251 \\ Z_2' &= Z_1'/V_2 = 0.6865 \\ Z_3' &= Z_2'\ V_3 = 1.764 \\ Z_4' &= Z_3'/V_4 = 0.604 \\ Z_5' &= Z_4'\ V_5 = 1.550 \\ Z_6' &= Z_5'/V_6 = 0.850 \\ Z_7' &= Z_6'\ V_7 = 1.065 \ \text{(output)} \end{aligned}\right\}. \tag{72}$$

It should be noticed that the output impedance, Z_7', of the half-wave filter is also the VSWR of the filter or transformer at center frequency [15] (Fig. 8).

In this example it was not necessary to interpolate from the tables for the V_i or the Z_i. When R is not given exactly in the tables, the interpolation procedure explained at the end of Section IV should be followed.

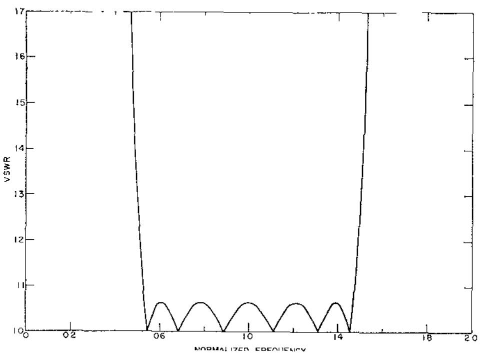

Fig. 8—Analyzed performance of transformer designed in Example 6.

VIII. Approximate Design When R is Large

Theory

Riblet's procedure [5], while mathematically elegant, and although it holds for all values of R, is computationally very tedious, and the accuracy required for large R can lead to difficulties even with a large digital computer. Collin's formulas [4] are more convenient (Section IV) but do not go beyond $n=4$ (Tables II to V). Riblet's procedure has been used to tabulate maximally flat transformers up to $n=8$ (Tables VIII and IX). General solutions applicable only to "small R" have been given in Sections VI and VII, and are tabulated in Tables VIII and X. In this part, convenient formulas will be given which become exact only when R is "large," as defined by

$$R \gg \left(\frac{2}{w_q}\right)^n. \tag{73}$$

These solutions are suitable for most practical *filter* applications (but not for practical *transformer* applications).

For "large R" (or small w_q), stepped impedance transformers and filters may be designed from low-pass, lumped-constant, prototype filters [2], [28], whose normalized reactive elements are denoted by g_i $(i=1, \cdots, n)$. The transformer or filter step VSWR's are obtained from

$$\left.\begin{aligned} V_1 &= V_{n+1} = \frac{4}{\pi}\frac{g_1\omega_1'}{w_q} \\ V_i &= \frac{16}{\pi^2}\frac{\omega_1'^2}{w_q^2}g_{i-1}g_i, \quad \text{when} \quad 2 \le i \le n \\ &(V_i \text{ large, } w_q \text{ small}), \end{aligned}\right\} \tag{74}$$

where ω_1' is the radian cutoff frequency of the low-pass prototype and w_q is the quarter-wave transformer fractional bandwidth [given by (1) for Chebyshev transformers and (9) or (10) for maximally flat transformers]. Again, the half-wave-filter bandwidth, w_h, is equal to one-half w_q [see (24)].

The V_i and Γ_i are symmetrical about the center in the sense of (41) and (42), although the g_i need not be similarly related.

With the tables of [28], it is easy to use (73). One should, however, always verify that the approximations are valid, and this is explained next. Procedures to be used in borderline cases, and the accuracy to be expected, will be illustrated by examples.

Range of Validity

The criteria given in (48) and (61) are reversed. The validity of the design formulas given in this part depends on R being large enough. It is found that the analyzed performance agrees well with the predicted performance (after adjusting R, if necessary, as in Examples 8 and 9) provided that (73) is satisfied; R should exceed $(2/w_q)^n$ by preferably a factor of about 10 or 100 or more (compare end of Section IX). The ranges of validity for "small R" and "large R" overlap in the region between (62) and (73), where both procedures hold only indifferently well. (See Example 9.)

For the maximally flat transformer, (73) still applies fairly well, when $w_{q,\,3\text{db}}$ is substituted for w_q.

As a rough but useful guide, the formulas of this section generally result in the predicted performance in the pass band when the pass-band maximum VSWR exceeds about $(1+w_q^2)$. This rule must be considered indeterminate for the maximally flat case $(w_q=0)$, when the following rough generalization may be substituted: The formulas given in this section for maximally flat transformers or filters generally result in the predicted performance when the maximally flat quarter-wave transformer 3-db fractional bandwidth, $w_{q,\,3db}$, is less than about 0.40.[5] The half-wave filter fractional bandwidth, $w_{h,\,3db}$, must of course be less than half of this, or 0.20.

After the filter has been designed, a good way to check on whether it is likely to perform as predicted is to multiply all the VSWR's, $V_1 V_2 \cdots V_{n+1}$, and to compare this product with R derived from the performance specifications using Table I and (13). If they agree within a factor of about 2, then after scaling each V so that their VSWR product finally equals R, good agreement with the desired performance may be expected.

Three examples will be worked out, illustrating a narrow-band and a wide-band design, and one case where (73) is no longer satisfied.

Example 7: Design a half-wave filter of 10-per cent fractional bandwidth with a VSWR ripple of 1.10, and with at least 30-db attenuation 10 per cent from center frequency.

Here $w_h = 0.1$, $\therefore w_q = 0.2$. A VSWR of 1.10 corresponds to an insertion loss of 0.01 db. From (33) and (31), or (17) and (12),

$$\mu_0 = \sin \frac{\pi w_h}{2} = \sin 9^\circ = 0.1564.$$

At 10 per cent from center frequency, by (32),

$$\frac{\omega'}{\omega_1'} = \frac{\sin \theta'}{\mu_0} = \frac{\sin 172^\circ}{0.1564} = 1.975.$$

From Fig. 2 of [2], or from [21], pp. 196 and 197, a 5-section filter would give only 24.5 db at a frequency 10 per cent from band-center, but a six-section filter will give 35.5 db. Therefore, we must choose $n = 6$ to give at least 30-db attenuation 10 per cent from center frequency.

The output-to-input impedance ratio of a six-section quarter-wave transformer of 20 per cent fractional bandwidth and 0.01-db ripple is given by Table I and (13) and yields (with $\mathcal{E}_r = 0.0023$ corresponding to 0.01-db ripple)

$$R = 4.08 \times 10^{10}. \tag{75}$$

Thus R exceeds $(2/\omega_q)^n$ by a factor of 4×10^4, which by (73) is ample, so that we can proceed with the design.

From [28], for $n = 6$ and 0.01-db ripple (corresponding to a maximum VSWR of 1.10), and from (74),

$$\left.\begin{aligned} V_1 = V_7 &= 4.98 \\ V_2 = V_6 &= 43.0 \\ V_3 = V_5 &= 92.8 \\ V_4 &= 105.0 \end{aligned}\right\}. \tag{76}$$

This yielded the response curve shown in Fig. 9, which is very close to the design specification in both the pass and stop bands. The half-wave filter line impedances are

$$\left.\begin{aligned} Z_0' &= 1.0 \quad \text{(input)} \\ Z_1' &= V_1 = 4.98 \\ Z_2' &= Z_1'/V_2 = 0.1158 \\ Z_3' &= Z_2'\ V_3 = 10.74 \\ Z_4' &= Z_3'/V_4 = 0.1023 \\ Z_5' &= Z_4'\ V_5 = 9.50 \\ Z_6' &= Z_5'/V_6 = 0.221 \\ Z_7' &= Z_6'\ V_7 = 1.10 \quad \text{(output)} \end{aligned}\right\}. \tag{77}$$

Note that $Z_7' = 1.10$ is also the VSWR at center frequency (Fig. 10).

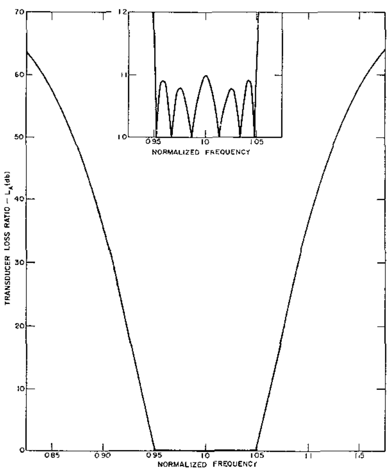

Fig. 9—Analyzed performance of half-wave filter designed in Example 7.

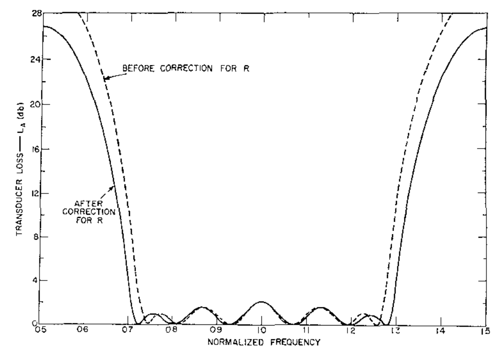

Fig. 10—Analyzed performance of two half-wave filters designed in Example 8.

[5] Larger 3-db fractional bandwidths can be designed accurately for small n, for example up to about $w_{q,\,3db} = 0.60$ for $n = 2$.

The corresponding quarter-wave transformer has a fractional bandwidth of 20 per cent; its line impedances are

$$\left.\begin{aligned} Z_0 &= 1.0 \quad \text{(input)} \\ Z_1 &= V_1 = 4.98 \\ Z_2 &= Z_1V_2 = 2.14 \times 10^2 \\ Z_3 &= Z_2V_3 = 1.987 \times 10^4 \\ Z_4 &= Z_3V_4 = 2.084 \times 10^6 \\ Z_5 &= Z_4V_5 = 1.9315 \times 10^9 \\ Z_6 &= Z_5V_6 = 8.30 \times 10^9 \\ R = Z_7 &= Z_6V_7 = 4.135 \times 10^{10} \text{ (output)} \end{aligned}\right\} \tag{78}$$

which is within about $1\frac{1}{2}$ per cent of R in (75). Therefore, we would expect an accurate design, which is confirmed by Fig. 9. The attenuation of 35.5 db at $f=1.1$ is also exactly as predicted.

Example 8: It is required to design a half-wave filter of 60 per cent bandwidth with a 2-db pass-band ripple. The rejection 10 per cent beyond the band edges shall be at least 20 db.

Here $w_h=0.6$, $\therefore w_q=1.2$. As in the previous example, it is determined that at least six sections will be required, and that the rejection 10 per cent beyond the band edges should then be 22.4 db.

From (13) and Table I it can be seen that, for an exact design, R would be 1915; whereas $(2/w_q)^n$ is 22. Thus R exceeds $(2/w_q)^n$ by a factor of less than 100, and therefore, by (73), we would expect only a fairly accurate design with a noticeable deviation from the specified performance. The step VSWR's are found by (74) to be

$$\left.\begin{aligned} V_1 = V_7 &= 3.028 \\ V_2 = V_6 &= 2.91 \\ V_3 = V_5 &= 3.93 \\ V_4 &= 4.06 \end{aligned}\right\}. \tag{79}$$

Their product is 4875, whereas from (13) and Table I, R should be 1915. The V_i must therefore be reduced. As in Example 6, we shall scale the V_i so as to slightly increase the bandwidth, without affecting the pass-band ripple. Since from (74) V_1 and V_{n+1} are inversely proportional to w_q, whereas the other $(n-1)$ junction VSWR's, namely $V_2, V_3, \cdots V_n$, are inversely proportional to the square of w_q, reduce V_1 and V_7 by a factor of

$$\left(\frac{1915}{4875}\right)^{1/2n} = \left(\frac{1915}{4875}\right)^{1/12} = 0.9251,$$

and V_2 through V_6 by a factor of

$$\left(\frac{1915}{4875}\right)^{1/n} = \left(\frac{1915}{4875}\right)^{1/6} = 0.8559.$$

(Compare Example 6.) This reduces R from 4875 to 1915. Hence,

$$\left.\begin{aligned} V_1 = V_7 &= 2.803 \\ V_2 = V_6 &= 2.486 \\ V_3 = V_5 &= 3.360 \\ V_4 &= 3.470 \end{aligned}\right\}. \tag{80}$$

The half-wave filter line impedances are now

$$\left.\begin{aligned} Z_0' &= 1.0 \quad \text{(input)} \\ Z_1' &= 2.803 \\ Z_2' &= 1.128 \\ Z_3' &= 3.788 \\ Z_4' &= 1.092 \\ Z_5' &= 3.667 \\ Z_6' &= 1.475 \\ Z_7' &= 4.135 \text{ (output)} \end{aligned}\right\}. \tag{81}$$

Since the reduction of R, from 4875 to 1915, is a relatively large one, we may expect some measurable discrepancy between the predicted and the analyzed performance. The analyzed performance of Designs (79) and (80), before and after correction for R, are shown in Fig. 10. For most practical purposes, the agreement after correction for R is quite acceptable. The bandwidth for 2-db insertion loss is 58 per cent instead of 60 per cent; the rejection is exactly as specified.

Discussion: The half-wave filter of Example 7 required large impedance steps, the largest being $V_4=105$. It would therefore be impractical to build it as a stepped-impedance filter; it serves, instead, as a prototype for a reactance-coupled cavity filter. This is typical of narrow-band filters. The filter given in the second example, like many wide-band filters, may be built directly from (80) since the largest impedance step is $V_4=3.47$ and it could be constructed after making a correction for junction discontinuity capacitances [3], [1]. Such a filter would also be a low-pass filter. (See Fig. 6). It would have identical pass bands at all harmonic frequencies, and it would attain its peak attenuation at one-half the center frequency (as well as at 1.5, 2.5, etc., times the center frequency, as shown in Fig. 6.) The peak attenuation can be calculated from (8) and (75). In Example 7 the peak attenuation is 100 db, but the impedance steps are too large to realize in practice. In Example 8 the impedance steps could be realized, but the peak attenuation is only 27 db. Half-wave filters are therefore more useful as prototypes for other filter-types which are easier to realize physically. If shunt inductances or series capacitances were used (in place of the impedance steps) to realize the V_i and to form a direct-coupled-cavity filter, then the attenuation below the pass band is increased and reaches infinity at

TABLE XI
THE THREE DESIGNS OF EXAMPLE 9

A—"Large R" Approximation.
B—"Small R" Approximation.
C—Exact Design.

	Design		
	A	B	C
$V_1=V_5$	1.656	1.780	1.936
$V_2=V_4$	2.028	2.091	1.988
V_3	2.800	2.289	2.140

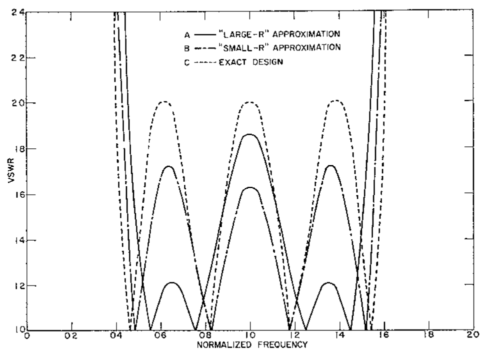

Fig. 11—Analyzed performance of three quarter-wave transformers designed in Example 9.

zero frequency; the attenuation above the pass band is reduced, as compared with the symmetrical response of the half-wave filters (Figs. 9 and 10). The derivation of such filters from the quarter-wave transformer or half-wave filter prototypes will be presented in a future paper.

Example 9: This example illustrates a case when neither the first-order theory (Section VI) nor the method of this part are accurate, but both may give usable designs. These are compared to the exact design.

It is required to design the best quarter-wave transformer of four sections, with output-to-input impedance ratio $R=31.6$, to cover a fractional bandwidth of 120 per cent.

Here $n=4$ and $w_q=1.2$. From (13) and Table I, the maximum VSWR in the pass band is 2.04. Proceeding as in the previous example, and after reducing the product $V_1V_2 \cdots V_5$ to 31.6 (this required a relatively large reduction factor of 4), yields Design A shown in Table XI. Its computed VSWR is plotted in Fig. 11 (continuous line, Case A).

Since R exceeds $(2/w_q)^n$ by a factor of only 4 [see (73)], the first-order procedure of Section VII may be more appropriate. This is also indicated by (62), which is satisfied, although (61) is not. Proceeding as in Example 6 yields Design B, shown in Table XI and plotted in Fig. 11 (dash-dot line, Case B).

In this example, the exact design can also be obtained from Tables IV and V, by linear interpolation of log V against log R. This gives Design C shown in Table XI and plotted in Fig. 11 (broken line, Case C).

Designs A and B both give less fractional bandwidth than the 120 per cent asked for, and smaller VSWR peaks than the 2.04 allowed. The fractional bandwidth (between $V=2.04$ points) of Design A is 110 per cent, and of Design B is 115 per cent, and only the exact equal-ripple design, Design C, achieves exactly 120 per cent. It is rather astonishing that two approximate designs, one based on the premise $R\approx 1$, and one on $R\to\infty$, should agree so well.

IX. ASYMPTOTIC BEHAVIOR AS R TENDS TO INFINITY

Cohn [2] developed formulas for direct-coupled cavity filters with reactive discontinuities. His formulas become exact only in the limit as the bandwidth tends to zero. This is not the only restriction. Cohn's formulas [2] for transmission-line filters, like our formulas in (74), hold only when (73) or its equivalent is satisfied. [Define the V_i as the VSWR's of the reactive discontinuities at center frequency; R is still given by (43); for w_q in (73), use twice the filter fractional bandwidth in reciprocal guide wavelength.] The variation of the V_i with bandwidth is correctly given by (74) for small bandwidths. These formulas can be adapted for design of both quarter-wave transformers and half-wave filters, as in (74) and hold even better in this case than when the discontinuities are reactive. This might be expected since the line lengths between discontinuities for half-wave filters become exactly one-half wavelength at band-center, whereas they are only approximately 180 electrical degrees long in direct-coupled cavity filters. (See Fig. 14 of [2].)

Using (74) and the formulas [2] for the prototype element values g_i ($i=1, 2, \cdots, n$), one can readily deduce some interesting and useful results for the V_i as R tends to infinity. One thus obtains, for the junction VSWR's of Chebyshev transformers and filters,

$$\left.\begin{aligned} \lim_{R\to\infty} V_i &= \left(\frac{8}{\pi w_q}\right)^2 \frac{\sin\left(\dfrac{2i-3}{2n}\pi\right)\sin\left(\dfrac{2i-1}{2n}\pi\right)}{\sin^2\left(\dfrac{i-1}{n}\pi\right)} \\ &= \left(\frac{4}{\pi w_h}\right)^2 \frac{\sin\left(\dfrac{2i-3}{2n}\pi\right)\sin\left(\dfrac{2i-1}{2n}\pi\right)}{\sin^2\left(\dfrac{i-1}{n}\pi\right)} \\ &= \left(\frac{4}{\pi w_h}\right)^2 \left[1 - \left\{\frac{\sin\dfrac{\pi}{2n}}{\sin\left(\dfrac{i-1}{n}\pi\right)}\right\}^2\right] \\ & (i = 2, 3, \cdots, n) \end{aligned}\right\}. \quad (82)$$

TABLE XII

TABLE OF $\left(\frac{w_q^2}{2}\right)\lim_{R\to\infty}(V_i)$ FOR SMALL w_q

$[V_i = V_{n+2-i}]$

n	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$
2	0.81056						
3	1.08075						
4	1.14631	1.38372					
5	1.17306	1.44999					
6	1.18675	1.47634	1.51254				
7	1.19474	1.48981	1.53668				
8	1.19981	1.49773	1.54885	1.55943			
9	1.20325	1.50282	1.55596	1.57073			
10	1.20568	1.50631	1.56052	1.57727	1.58146		
11	1.20747	1.50880	1.56365	1.58145	1.58762		
12	1.20882	1.51066	1.56589	1.58431	1.59153	1.59351	
13	1.20987	1.51207	1.56757	1.58636	1.59419	1.59723	
14	1.21070	1.51318	1.56886	1.58789	1.59610	1.59975	1.60081

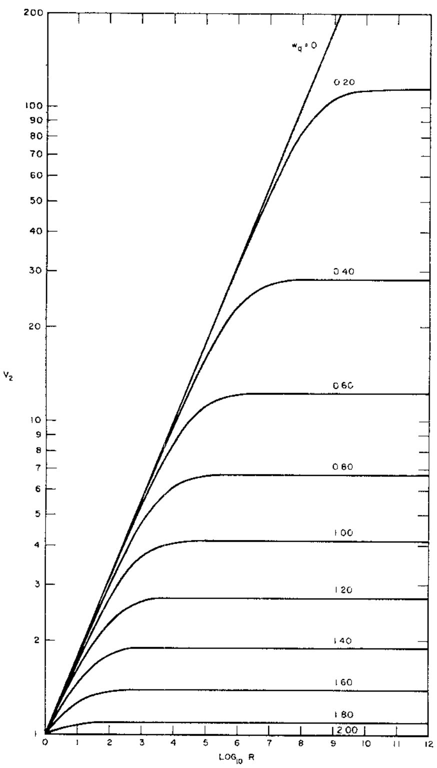

Fig. 12—V_2 vs log R of four-section transformer for all fractional bandwidths in steps of 0.20.

The quantity

$$w_h^2 \lim_{R\to\infty} (V_i) = \left(\frac{w_q}{2}\right)^2 \lim_{R\to\infty} (V_i) \tag{83}$$

is tabulated in Table XII for $i=2, 3, \cdots, n$ and for $n=2, 3, \cdots, 14$.

We notice that for Chebyshev transformers and filters, the V_i $(i\neq 1,\ n+1)$ tend to finite limits, and thus $V_1=V_{n+1}$ tend to a constant times $R^{1/2}$. We also see that

$$w_h^2 V_i < \frac{16}{\pi^2} = 1.62115 \qquad (i = 2, 3, \cdots, n) \tag{84}$$

for all n, and tends to $16/\pi^2$ only in the limit $i\to n/2\to\infty$.

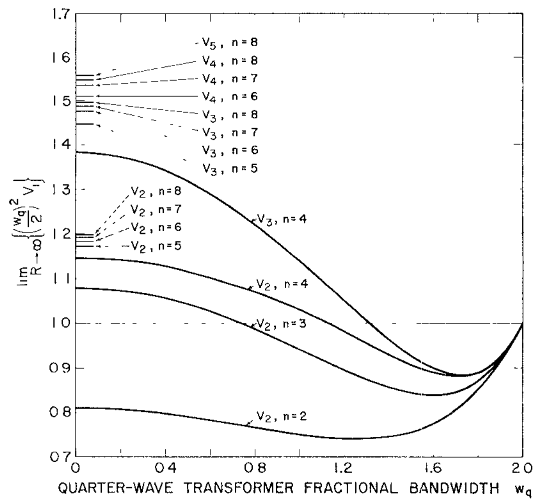

Fig 13—$\text{Lim}_{R\to\infty}(w_q/2)^2 V_i$ plotted against fractional bandwidth for transformers having up to four sections, and shown for small w_q up to eight sections.

For maximally flat transformers, the V_i all tend to infinity with R, but the quantities

$$A_1 = A_{n+1} = \frac{V_1}{R^{1/2n}}$$

$$A_i = \frac{V_i}{R^{1/n}} \qquad (i = 2, 3, \cdots, n) \tag{85}$$

tend toward finite limits given by

$$\left.\begin{aligned} \lim_{R\to\infty} A_1 &= 2^{(n-1)/n} \sin\left(\frac{\pi}{2n}\right) \\ \lim_{R\to\infty} A_i &= 2^{2(n-1)/n} \sin\left(\frac{2i-1}{2n}\pi\right)\sin\left(\frac{2i-3}{2n}\pi\right) \end{aligned}\right\} \tag{86}$$

from which we see that

$$\left.\begin{aligned} V_1 = V_{n+1} &< \left(\frac{4^{n-1}}{R}\right)^{1/2n} \\ V_i &< \left(\frac{4^{n-1}}{R}\right)^{1/n} \end{aligned}\right\} \tag{87}$$

for all n. They tend toward the values on the right-hand side only in the limit $i \to n/2 \to \infty$.

To show how a typical V_i approaches its asymptotic value, the exact solution for V_2 when $n=4$ is plotted in Fig. 12 for all fractional bandwidths w_q in steps of 0.20. It is seen that each curve consists of two almost linear regions with a sharp knee joining them. In the sloping region above the origin ("small R"), the approximations of Sections VI or VII apply; in the horizontal region ("large R"), the approximations of Section VIII apply. These two sets of approximations probably hold as well as they do because the knee region is so small.

The exact asymptotic values of $w_h^2 V_i = (w_q/2)^2 V_i$ are plotted against w_q in Fig. 13. If (82) were exact instead of approximate, then all of the curves would be horizontal straight lines. As it is, (82) gives the correct value only on the $w_q=0$ axis. As the bandwidth increases, $w_h^2 V_i$ departs from the value at $w=0$ slowly at first, then reaches a minimum, and finally all curves pass through unity at $w_q=2(w_h=1)$. The values of $(w_q/2)^2 V_i$ at $w_q=0$ up to $n=8$ are also shown in Fig. 13. They all lie below the value $16/\pi^2=1.62115$, and may be expected to exhibit the same sort of general behavior as do the curves up to $n=4$, for which the exact solutions were obtained from (37) to (39).

The asymptotic values of the V_i for $i=2, 3, \cdots, n$, and for a given fractional bandwidth, are seen to be fairly independent of n, on examination of (82), Table XII, or Fig. 13. It follows that the same is true of $V_1/\sqrt{R}=V_{n+1}/\sqrt{R}$. Thus, as R increases indefinitely, so do V_1 and V_{n+1}; on the other hand for "small R," V_1^2 and V_{n+1}^2 are less than the other V_i (*not* squared) for small and moderately wide fractional bandwidths (up to about 100 per cent bandwidths, by Table VIII). If we assume that in the knee region (Fig. 12) $V_1^2=V_{n+1}^2$ are of the order of the other V_i, then in the knee region R is of the order of $(V_i)^n$, for any $i \neq 1, n+1$. From (74), R is therefore inversely proportional to (const.$\times w_q^2)^n$, and from the previous remarks this constant of proportionality is reasonably independent of n. Using Fig. 12 for example, the constant is very close to the value $\frac{1}{2}$. This leads to the magnitude formulas of (48), (61), and (62), and (73), which have been confirmed by numerous sample solutions.

X. Conclusion

The theory of the quarter-wave transformer has been reviewed and extended, and the major results have been presented. The distinctions between ideal and nonideal junctions, homogeneous and inhomogeneous transformers, synchronous and nonsymchronous tuning, have been brought out explicitly. The concept of half-wave filters will be found useful in the design of direct-coupled-cavity filters. Design formulas were presented, numerical tables were given, methods of numerical calculation were explained, and their use was illustrated by several examples. Where exact solutions are not available, two approximate design procedures were given: one is applicable where R is small enough, and one where R is large enough. To help in obtaining accurate numerical solutions, the connection between antenna arrays and "small-R" transformers was utilized, as was the connection between lumped-constant, low-pass filters and "large-R" transformers.

An additional bibliography lists related topics not covered in this paper.

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In-Line Waveguide Calorimeter for High-Power Measurement*

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Summary—**The static in-line calorimeter measures the temperature rise in the walls of a waveguide caused by the attenuation of microwave power flowing through the waveguide. It is simple and inexpensive and can be constructed so that it will fit on waveguide already existing in a microwave system. The device should be reliable because it uses no active circuitry. In addition, few mechanical problems are encountered in its use because the existing waveguide need not be altered. The theory of the device is developed, and two experimental *S*-band calorimeters using stainless steel waveguide and resistance-wire bridge temperature indicators are described. The measured sensitivity and time constant for both units fall within the experimental error of confirming the theoretically predicted figures.**

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Introduction

The HIGH-POWER measurement or monitoring schemes presently available, if not complete absorption devices, are usually reduced-signal sampling devices in which a low-power meter is used. In most low-level measurement schemes, however, the background noise (ambient temperature fluctuations in the case of the bolometer power meter) often determines the ultimate resolution of the device. In a high-power measurement system, on the other hand, the power level present most often completely masks the low-power background noise. It may then be less desirable to sample a high-power signal through a directional coupler and to attenuate the sampled signal until it can be read with conventional low-power meters than to use a